TEACHING

Mathematics

IN THE PRIMARY SCHOOL

G. D. BISHOP, M.A., B.SC., PH.D.

COLLINS

9391

9/6d
For.

9391

STATE INSTITUTE OF EDUCATION ★ BANIPUR ★

TEACHING MATHEMATICS IN THE PRIMARY SCHOOL

STATE INSTITUTE OF EDUCATION ★ BANIPUR ★

Also by Dr. Bishop

TEACHING MATHEMATICS
IN THE SECONDARY SCHOOL

TEACHING
Mathematics
IN THE PRIMARY SCHOOL

GEORGE D. BISHOP
M.A., B.Sc., Ph. D., Dip.Ed.

Senior Lecturer in Education and in Methods
of Mathematics Teaching, University
of the West Indies

STATE INSTITUTE OF EDUCATION ★ BANIPUR ★

S.C.E.R.T. West Bengal ★ Library Calcutta 25/3, B. C. Road ★

Collins

LONDON AND GLASGOW

S.C.E.R.T., W.B. LIBRARY
Date 6.5.2008
Accn. No. 13351

First edition, 1964
© George D. Bishop, 1964

Printed in Great Britain at the Press of
the Publishers, Cathedral Street, Glasgow

Preface

THIS BOOK attempts to bring together under one cover those invaluable suggestions and hints on the enlightened and successful teaching of Mathematics in the Primary School that have been proposed from time to time by inspectors, educationists and other experts.

This book is offered to those many teachers who, owing to the great shortage of qualified mathematics teachers, find themselves teaching a subject for which they have had no proper training and, perhaps, even for which they have no particular relish. Others just beginning their careers who, because of their inexperience, look forward to teaching so important a subject as mathematics with trepidation and misgiving will here, it is hoped, find the guidance and advice necessary to lead them forward with greater confidence. In the compiling of this book the needs of students in Training Colleges have been kept in mind. Finally, it is hoped that even experienced mathematics teachers will find something of interest and of value in the ensuing pages.

Acknowledgements

The Author wishes to acknowledge his indebtedness to the following books which were among some of those used for reference:

ADAMS, L. D. *A Background to Primary School Mathematics* (Oxford University Press, 1953).

BALLARD, P. B. *Teaching the Essentials of Arithmetic* (University of London Press, 1945).

GAGG, J. C. *Common Sense in the Primary School* (Evans Bros., London, 1951).

PIAGET, J. *The Child's Conception of Number* (Routledge & Kegan Paul, Ltd., London, 1952).

SCHONELL, F. J. *Diagnosis and Remedial Teaching in Arithmetic* (Oliver and Boyd, Edinburgh, 1958).

SUMNER, W. L. *The Teaching of Arithmetic and Elementary Mathematics* (Blackwell, Oxford, 1955).

Mathematical Association. *The Teaching of Mathematics in Primary Schools* (G. Bell & Sons Ltd., 1955).

Scottish Council for Research in Education. *The Teaching of Arithmetic*: Publication XXI; *Addition and Subtraction Facts and Processes*: Publication XXVIII; *Studies in Arithmetic*, Vol. I, II (University of London Press).

The Author and Publishers wish to thank Miss L. D. Adams for permission to use the extract quoted on page 55, taken from page 96 of *A Background to Primary School Mathematics*, published by Oxford University Press in 1953.

Contents

INTRODUCTION 9
FIRST NOTIONS OF NUMBER 13
FIRST NOTIONS OF QUANTITY, SIZE AND SHAPE 19
THE WRITTEN LANGUAGE OF ARITHMETIC 21
THE BASIC PROCESSES 27
  Addition
  Subtraction
  Multiplication
  Division
DRILL 49
PROJECTS AND ACTIVITIES 52
THE BASIC PROCESSES WITH MONEY 54
  Compound Addition
  Compound Subtraction
  Compound Multiplication
WEIGHTS AND MEASURES 58
  Length
  Weight
  Capacity
  Time
ORAL AND MENTAL ARITHMETIC 63
  Short Cuts
FRACTIONS 65
DECIMAL FRACTIONS 71
THE BEGINNINGS OF GEOMETRY 73
ANGLES 76
AREA 77
VOLUME 81
GRAPHS 82
APPENDICES 85
  Primary Mathematics Experiments in the U.S.A.
  Developments in Primary Mathematics in Britain
  The Cuisenaire Method

Introduction

THE TRADITIONAL methods of education placed the emphasis on teaching and so were centred on the teacher. Whilst the teacher and his teaching are very important factors, modern methods of education emphasise more the learning process, which means the education is child-centred. The child is led to build up understanding through his own active occupations, through his own experiments, explorations and discoveries, be they even in play. Instead of mathematics being one long boring mechanical grind with numbers and figures and meaningless formulæ, it must be made full of fun and interest to the children, stimulating and invigorating, not dull and lifeless. If mathematics is to be made so, the children must be provided with an abundance of real-life, not formal, situations and opportunities which will prove more of a challenge to their intelligence and less to their memory. They should discover for themselves, instead of being bent under a mass of uncomprehended results and rules.

One of the most effective learning devices is project work. Since in a project the children learn through their own discoveries arising out of their immediate needs, the project is far superior to a purely artificial exercise in the classroom. It should be one of the principal tools of the teacher. Whenever possible a project should involve the children searching out information for themselves, making use of it—preferably in groups so as to foster the spirit of co-operative effort—in the building or construction of something purposeful, and if possible and the occasion demands it, an excursion or visit so that the children get some further insight into "mathematics in action". Through activities such as these children are made to see that mathematics can be —and is—meaningful, useful, and interesting. They will be doing mathematics almost without realising it.

In the early years it is even more essential that the work done be active and realistic; that the children learn through direct personal experience. In fact, unless there is direct personal experience, what ideas the children acquire will be confused and without meaning. In the Infant school there should be a minimum of written work, though, of course, there is no objection to letting children draw and label what they have done or "discovered". By such means they will reveal their understanding of what they have seen or done.

To ensure that the whole class are gainfully employed the teacher must provide sufficient opportunity and ample material or apparatus. To obviate the need for large quantities of apparatus, the teacher should, whenever possible, divide his class into groups, with different groups using different items. Most children enjoy working in groups, and the arrangement is usually very satisfactory from an educational point of view since the weaker pupils derive stimulus and confidence from their abler fellows, whilst the latter gain in insight from the experience of elucidating the difficulties of their weaker companions. This arrangement will also be found to work admirably in large classes containing children of very varied ability and attainments.

This being the way to teach mathematics, it implies that the teacher should no longer hold the centre of the stage. It is for him to provide his pupils with such stimulating opportunities as will increase their mathematical perception and understanding through *their* own purposeful activities. Standing unobtrusively in the wings, the teacher will guide the various operations, answer the children's questions, make suggestions, ask just the right question at the critical point in the child's argument. He will supply new words such as *area*, etc., when the children are trying to describe their discoveries, and ensure that the children's experiences are not left vague and unco-ordinated.

Finally, of course, the teacher must remember that his manner, voice, behaviour and his relations with his class have as much influence as his methods of teaching. Since emotional conditions can and do influence mathematical ability to a considerable extent, the teacher must aim to cultivate a sympathetic, encouraging and stimulating attitude towards his young charges; stern censure and harshness should be absent from his classroom. Dr. Schonell mentions in his *Diagnosis and Remedial Teaching in*

Arithmetic, 1958, p. 32, the case of the teacher who rarely kept quiet for more than a few minutes, who constantly upbraided his pupils, who censured all lapses in calculation, who demanded 100 per cent accuracy, who marked sums wrong with large crosses whilst small, scarcely visible, ticks marked those right, who never gave a word of praise or of encouragement. No wonder this teacher's headmaster complained that the class knew *less* mathematics than they did before!

First Notions of Number

THE FIRST THING a child must learn in arithmetic is counting. The whole of arithmetic rests on number, and until the child can count he can do nothing else. Giving a child formal lessons in arithmetic before he can count accurately is a complete and utter waste of time.

A young child gains his first notions of number through his eyes and fingers; through playing with and handling actual objects. In working out the answers to incidental questions such as "How many chairs do we want?" etc.; in his games of jumping, skipping, clapping, etc., a certain number of times, he slowly builds up his perception of numbers. He counts runs in cricket, bananas, coconuts, matchsticks, etc.; he arranges them in groups and in varying patterns; and from this "doing", his ideas of number slowly develop. Furthermore, the meanings and concepts he acquires of the simpler numbers and of their relationships, having been gained by direct personal experience, are real, clear and accurate. By "doing", visual imagery is evoked to assist the children's reasoning. In brief, he begins to make "sense out of sensation".

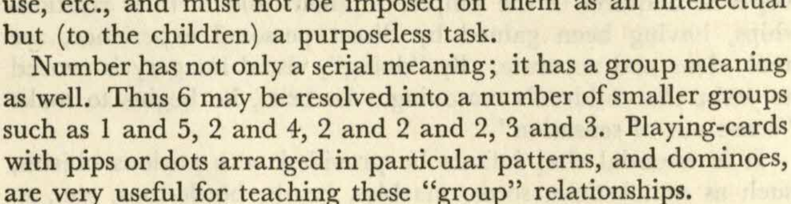
It is essential that infants be provided with various objects, such as matchsticks, seeds, marbles, beads, bottle-tops, etc., so that they have ample opportunity to find out *for themselves* the relations between numbers. A warning here is necessary. In teaching number—as, indeed, in teaching anything—we cannot force the pace. Most of the difficulty in arithmetic arises because teachers hurry their pupils too much in the early stages. The knowledge of number and the ability to perform number operations come with mental growth which proceeds naturally in its own time. Trying to accelerate this growth prematurely will do more harm than good: real progress demands that children make haste slowly. So often a teacher rushes on to teaching skills, such as formal addition, when his pupils have still not acquired a foundation of wide practical experience. The result is that the children acquire spurious and superficial techniques of which they have no understanding whatsoever. And understanding must precede drill and formal exercises. So what is absolutely imperative is that the teacher provide his young pupils with an abundance of opportunities for them to acquire this practical experience, using the materials of experience, which are those of ordinary living and of common childlike occupations and interests.

The teacher must be alert against the tendency to assume that because a child can recite "one, two, three, four", etc., he has mastered the relationships of number. The way to find out if the child has really done so is by asking him to fetch three books from a shelf, asking him the number of panes in a window, how many boys (or girls) there are in the class and so on. Whilst such questions are asked principally to give the teacher some notion of the child's progress, it must not be forgotten that for young children counting must be related to some definite purpose of *their* own, such as setting out chairs for an audience, checking the number of children in class, cans for milk, inkwells for class use, etc., and must not be imposed on them as an intellectual but (to the children) a purposeless task.

Number has not only a serial meaning; it has a group meaning as well. Thus 6 may be resolved into a number of smaller groups such as 1 and 5, 2 and 4, 2 and 2 and 2, 3 and 3. Playing-cards with pips or dots arranged in particular patterns, and dominoes, are very useful for teaching these "group" relationships.

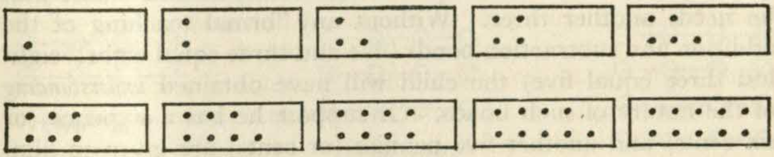
Games such as building or matching patterns, finding which card or domino is missing, naming the various patterns, and others, will result in children learning about number informally.

The domino system is particularly useful since it falls in naturally with our decimal notation, as it makes the half of ten the unit of design, with ten as a sort of double unit.



These patterns also bring out the relationship between one number and the next; as, for instance, the ⁙ includes the domino ⁘ and one more dot. Another advantage of the domino system is that it shows at a glance by how much the numbers fall short of ten.

Although patterns play an important part in learning about numbers, it is unwise to allow one type of pattern to become dominant. The Montessori system of grouping by twos can also be used.



Sticks tied up in bundles of ascending order of magnitude may also be used to teach number relationships.

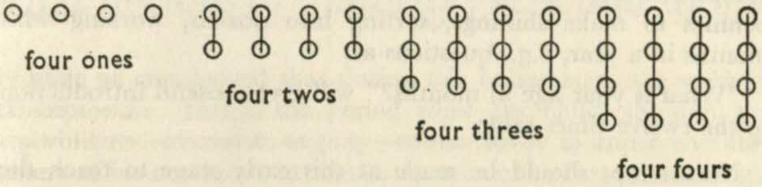
Even when they are adept at counting, it takes young children a long time to assimilate the idea that a number such as fourteen consists of two parts, four and ten. Before reaching this stage they will have needed considerable practical experience in putting two small groups (of dominoes, bundles of sticks, etc.) together, e.g., four and three, five and two, etc.

Without any formal teaching of addition, the child will gradually have realised from his experiences with various objects that a group such as five consists of two or more parts, namely four and one, three and two. If he has 3 pennies (or cents) and is given another four, he will realise that his wealth is seven pennies (or cents). Letting pairs of children put together the results of their "work" or their scores in games will give further opportunities for them to acquire a glimmering of the concept of addition. Buying two items at the class "shop" also provides opportunities for adding simple numbers.

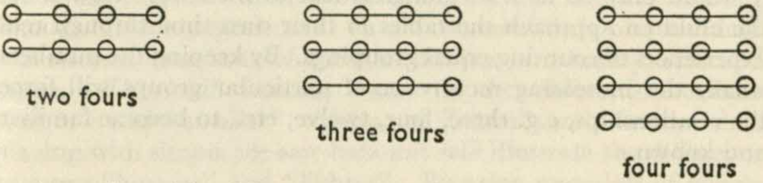
It must not be forgotten that whilst to the adult mind the addition of six and five is the same as that of five and six, the young child does not realise that it does not matter which group is counted first. It is not self-evident to young children that since seven pennies and five pennies make a shilling, five pennies and seven pennies must also make a shilling. They only learn this from experience. The class shop will provide numerous situations that will help children to become conscious of the concept of addition.

Situations such as the following will introduce young children to the notion of subtraction. John has three pennies (or cents), but the ball in the shop costs eight pence (or cents). Slowly it dawns on him that he will need another five pennies (or cents). When he has collected five pennies (or cents) he will realise that he needs another three. Without any formal teaching of the addition and subtraction bonds (five and three equal eight; eight less three equal five) the child will have obtained *understanding* of the nature of such bonds. Or suppose he has a sixpence (or six cents) and another five pennies (or cents) are given to him, a question such as whether he has enough money to buy a toy marked 10 pence (or cents) will give him further insight into an understanding of number bonds. Or when the child sees he has eight marbles but that his friend has fifteen, he will wonder how many marbles fewer he has and may become aware of the "$8+7=15$, $15-8=7$" bond. Rods marked into equal sections are also very useful to show simple addition relations such as seven and four equal eleven which equals ten and one. They are equally effective in establishing the complementary subtraction bonds. Through many other rich experiences, such as counting stock after sales in the class shop and working out how much is needed to re-stock the shop, or giving change after a purchase (up to a limit of 2 shillings or 20 cents) the child will become familiar with the number bonds. In course of time these bonds will become memorised to the point of automatic response. But once again it must be pointed out that for most children number relationships need a very long time before they are sufficiently understood for mechanical response.

Through his experiences the child will gradually acquire notions not only of addition and subtraction but also of multiplication and division. Thus, finding by successive addition the cost of a pile of two-penny (or two-cent) bus-tickets will give the child the sequence of twos; counting threepenny pieces as pence brings out the series of threes and so on. Arranging counters as follows will give the four-times table:



If the counters are arranged thus:



we have the table of fours.

Questions such as:

"There are 15 desks in the class and at each desk are two children. How many children are there in the class?"

"How many legs have twelve cats?" etc., etc.

will give further experience of the *meaning* of multiplication. Children making cards of buttons, say twelve each, for the class "draper's shop" will sew them on in three rows of four or four rows of three. This practical experience will enable them to comprehend the three, four, twelve relationship much more soundly than those who have learnt from rote repetition that three fours are twelve.

Questions such as:

"Six wheels—how many bicycles?"

"100 legs—how many people?"

"Twenty-four wheels—how many cars?"

will introduce the child to simple division. When children themselves carry out the practical experiments of sharing, say, fifteen sweets among five children, when they "discover" that each gets three sweets, and also that three sweets taken five times makes a total of fifteen, they will begin to understand the complementary nature of multiplication and of division. Piling pennies to make shillings, sorting into dozens, working with months in a year, e.g., questions as:

"What is your age in months?" will give a useful introduction to the twelve times table.

No attempt should be made at this early stage to teach the multiplication tables. If the children did manage to learn them, it would only be as a meaningless feat of memory. Rather let the children approach the tables in their own time through real experiences of counting equal groupings. By keeping the numbers small, the increasing recurrence of particular groups will force the relationships, e.g. three, four, twelve, etc., to become familiar and known.

First notions of quantity, size and shape

IT MUST BE emphasised that during this Infant stage the work is all kept *oral*. This is the period when the child, through his experiments—of course, as play—comes slowly to apperceive the nature of his environment and to assimilate the first glimmerings of the elementary concepts of mathematics. This is the period when the teacher must concentrate on the child's learning through his *own* experiences. In the course of their play children will become conscious of the first notions of quantity, size and shape. They will become aware of the difference between "larger" and "smaller" objects, of "more" and "less", while playing with simple see-saw balances will illustrate the difference between "heavier" and "lighter". By using materials of widely differing densities, such as marbles and peas, the children will come to "see" the possibility of balancing a small *bulk* with a larger one.

Playing with mosaics, jig-saws, bricks, boxes and cylindrical tins which can be packed one inside the other will give the young child his early experiences of size and shape and space. A lump of clay or Plasticine rolled out into a long snake or flattened into a pancake or modelled into various shapes can give the child his first geometrical experiences. The teacher should use words such as *square*, *ring*, *circle*, *corner*, *curve* when the context makes what is meant quite clear to the children. The teacher must be on his guard against using words before the child has had the experience necessary to comprehend the meaning. A safe plan is to let the child talk about his "discoveries", the teacher supplying the new word as and when the need for it arises.

Playing with water, or dry sand, using vessels of a variety of shapes and sizes, will not only fascinate young children and keep them amused for hours, but will also provide useful and fundamental mathematical notions, admittedly vague. The advantage of children's experimenting with water or dry sand is that various quantities can be broken up and then put together again. Of course, a great stumbling block to a child is that when he pours out liquid from a bottle or container into a new container, it looks quite a different amount. Children must learn this idea of sameness of capacity with difference of shape only from experience.

Experience of a few common measures, with computation still at the oral level, is not only appropriate to a young child's activities but can also be a very fruitful preliminary to written arithmetic. His playing at "shop" will have made him familiar with the value of coins and the different parts of the shilling and pound, or the dollar. In course of time he will want to compare one quantity with another, and slowly the idea of a *unit* will emerge. Of course, his units will be based on his own hands, or feet or strides. Soon also will emerge his appreciation that, for a given distance, if the unit is a larger one, the count will be fewer.

The foregoing pages have emphasised that a great variety of experience must precede any formal work in arithmetic. The first notions of number, the complementary nature of addition and subtraction, of multiplication and division, the first notions of quantity, shape and size, must all be rooted in experience. The task of the teacher is to provide this rich experience.

S.C.E.R.T., W.B. LIBRARY
Date 6.5.2008
Acc. No. 13351

The Written Language of Arithmetic

No WRITTEN arithmetic should be attempted *until* the children have acquired familiarity with the corresponding mental processes. Their learning must be based on adequate experience, from which will follow understanding. And it must never be forgotten that this understanding comes slowly. Through concrete experiences and oral discussion the children will have become familiar with the language of arithmetic. They will have made oral statements describing their mental processes in the course of their play, e.g.:

"I had seven sweets. I ate three. I have four left."

The next stage will be the writing down of their thoughts. Now is the time to introduce them to *figures*. And as figures are easier to write than the corresponding words, they will quickly come to use them in their written statements, e.g. 7 sweets. I ate 3. I have 4 left.

The signs +* and − can be introduced so that eventually the child can express his concepts as $7-3=4$. Again, a warning here will be timely. If the symbols +, −, × and ÷ are introduced prematurely, i.e. before the children have appreciated number relationships arising from a wealth of practical experience, they will be quite meaningless to the children. It is not till they are eight or nine years of age that children can correctly use words like *add*, *subtract*, *multiply* and *divide*.

There are many interesting ways of helping young children to learn and to consolidate the use of figures through playing with various devices. See the top diagram on p.22.

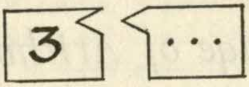
Flash cards carrying dots can be used very successfully in teaching number. By making children place figures alongside the flash cards, or, conversely, placing the cards alongside the

---

*The word "plus" is short for "surplus". In medieval warehouses the signs + or − were chalked on sacks, barrels, etc., to signify excess or short weight.

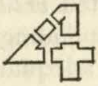
9391
~~9397~~

Jigsaw puzzles

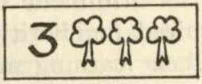


Peg figures

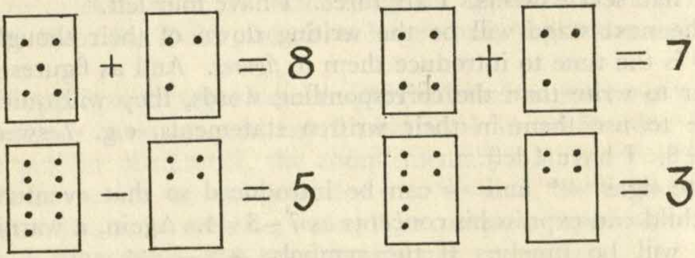
Puzzle figures



Number pictures

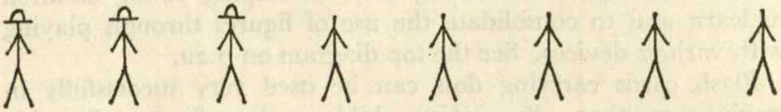


numbers, they will come to associate the figures with the respective groupings. Later they will use two cards for numbers larger than ten. Thus sixteen will be represented by the ten card and the six card. These cards can also be used to give simple sums in addition and subtraction, e.g.:



Interesting variations in teaching early notions of number and of simple addition and subtraction will be problems such as the following:

"Draw eight men standing in a line. Put hats on three of them. How many men have no hats?"



The first chart on p. 23 will be found useful for practice in the early basic addition facts.

The children having been shown how to use the chart by moving along the lines, e.g. $0+0=0$; $0+1=1$; $0+2=2$; and so on, will be able to practise with all the addition facts up to

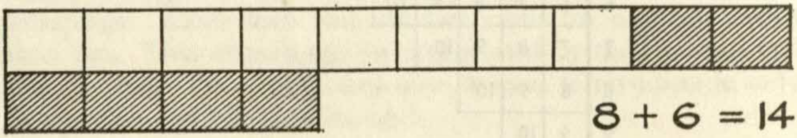
	0	1	2	3	4	5	6	7	8	9	10
0	0	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10	
2	2	3	4	5	6	7	8	9	10		
3	3	4	5	6	7	8	9	10			
4	4	5	6	7	8	9	10				
5	5	6	7	8	9	10					
6	6	7	8	9	10						
7	7	8	9	10							
8	8	9	10								
9	9	10									
10	10.										

10. The chart can also be used by pairs of children, one asking the question, the other giving the answer.

The following is the corresponding complementary subtraction chart for numbers up to 10.

	1	2	3	4	5	6	7	8	9	10
0	1	2	3	4	5	6	7	8	9	10
1	0	1	2	3	4	5	6	7	8	9
2		0	1	2	3	4	5	6	7	8
3			0	1	2	3	4	5	6	7
4				0	1	2	3	4	5	6
5					0	1	2	3	4	5
6						0	1	2	3	4
7							0	1	2	3
8								0	1	2
9									0	1
10										0

As a result of their play and experiments children will have acquired quite a wide range of knowledge concerning number relationships. They should be able to deal with combinations of less than ten up to nine plus nine. For some children it may be necessary to teach the addition table by getting them to arrange equal strips of paper in various number combinations. Thus to add 8 and 6, two squares are attached to the eight squares to make up a complete row of ten squares whilst the remaining four squares are separated and form part of the next row, thus:



They should know these combinations without even stopping to think; thus seeing 8 and 6 the children should think "fourteen" as a single pulse of thought.

Eventually all the simple number bonds must be learned and known thoroughly in all their forms, e.g.

$7+6=13$, $6+7=13$, $13-7=6$, $13-6=7$.

The subtraction facts should *not* be taught separately; they should be taught in association with the corresponding addition facts with which they are associated. Eventually, bonds such as $3+5=8$ will be extended to $13+5=18$, $23+5=28$ and so on.

Practice in addition can be given by writing a number, say 8, on the board, and asking the children to add repeatedly another number, say 9, thus giving totals of 17, 26, etc.; similarly for subtraction. Various number games may also be employed to help speed of response.

The calendar provides a useful source of number practice. If Jan. 1st is a Saturday, then the Saturdays in January will be the 8th, 15th, 22nd, 29th. Mondays will be the 3rd, 10th, 17th, and so on.

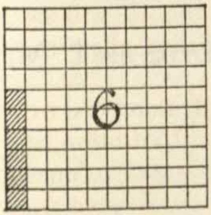
Only when the addition and subtraction of small numbers are securely known should the class proceed to written addition of tens and units, otherwise the habit of adding by counting in ones will be perpetuated.

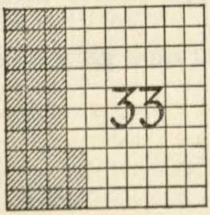
1	11	21	31	41	51	61	71	81	91
2	12	22	32	42	52	62	72	82	92
3	13	23	33	43	53	63	73	83	93
4	14	24	34	44	54	64	74	84	94
5	15	25	35	45	55	65	75	85	95
6	16	26	36	46	56	66	76	86	96
7	17	27	37	47	57	67	77	87	97
8	18	28	38	48	58	68	78	88	98
9	19	29	39	49	59	69	79	89	99
10	20	30	40	50	60	70	80	90	100

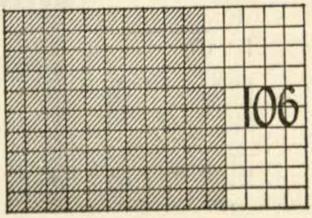
The pattern of the intervals of ten shown in the above chart helps to build up a mental picture of the notation and of the number sequence. The pupils themselves should make these charts; in the making the number pattern will become apparent. Training the children to read the chart not only from top to bottom and from left to right, but also from bottom to top and from right to left, helps in preparing the ground for the complementary nature of addition and subtraction.

The rhythmic pattern of the sequence of the numbers is also brought out when, in counting, emphasis is put on successive decades, e.g. "*twenty*, twenty-one, . . . *thirty*, thirty-one", etc. Children very soon see that the figures 1 to 10 are repeated, first with a unit interval, then with a ten-interval.

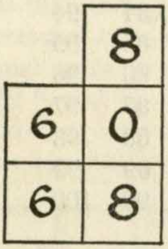
The idea of place value, viz. that figures have significance according to their position in the number, is a difficult one for children. It takes time for them to realise that in a number written as, say, 14, the 1 has the value of ten. Representing numbers graphically as shown will prove of value in teaching place-value.







Children should be asked to shade or colour graph paper squares to represent given numbers.



Another useful device is to have number cards. To illustrate the number 68 the teacher would select the 60 and the 8 cards and then place them upon each other as shown.

Numbers such as 176 should always be read as "one hundred and seventy-six", and never as "one seven six". Using a number such as, say, 176 the children should be asked to experiment with the same digits so as to make the largest and smallest numbers with the given digits.

The Basic Processes

Addition

Of course, in Arithmetic—as in all subjects—the sort of exercises set for the children should begin with the simple, gradually building up to other more difficult ones involving additional notions of the number concept. The following is such a graded set of exercises.

(i) Children find it easier to deal with vertical addition. They take less time and are more accurate. Even so, they should be given occasional practice in horizontal addition: 3+5, 6+3, etc.

$$\begin{array}{rr} 3 & 6 \\ +5 & +3 \\ \hline \\ \hline \end{array}$$

(ii) Adding several numbers of one digit is a useful preliminary to the addition of numbers with two digits. The children will find plenty of opportunities for such additions in adding small scores in games, etc.

$$\begin{array}{r} 3 \\ 4 \\ 5 \\ \hline \\ \hline \end{array}$$

(iii) Addition with tens and units will arise in such situations as trying to find the total class attendances for two or more classes, or for two or more days. Addition with tens and units, or even hundreds, tens and units, first should involve no carrying:

$$\begin{array}{rr} 36 & 223 \\ 41 & 456 \\ \hline \\ \hline \end{array}$$

(iv) Then exercises carrying units to tens:

$$\begin{array}{rr} 38 & 234 \\ 46 & 128 \\ \hline \\ \hline \end{array}$$

Then carrying tens to hundreds:

$$\begin{array}{rcr} 362 & \quad & 173 \\ 284 & & 456 \\ \hline \\ \hline \end{array}$$

Then carrying units to tens and tens to hundreds:

$$\begin{array}{r} 786 \\ 147 \\ \hline \\ \hline \end{array}$$

(v) Addition with numbers containing zero:

$$\begin{array}{rcrcrcrcr} 440 & \quad & 303 & \quad & 70 & \quad & 307 & \quad & 396 \\ 36 & & 424 & & 420 & & 403 & & 204 \\ \hline \\ \hline \end{array}$$

Research has shown that children make most errors in addition when a nought is involved.

(vi) Finally, addition involving more than two numbers:

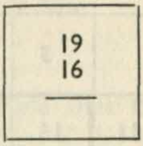
$$\begin{array}{r} 36 \\ 43 \\ 28 \\ 14 \\ \hline \\ \hline \end{array}$$

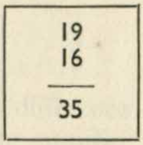
The children should be taught to add upwards first, and to check the answer by adding from the top downwards. In adding the child should say mentally, in this sum, for instance, "four, twelve, fifteen, twenty-one". The one is written down and the two carried mentally (or written at the bottom of the column to which it belongs).

It is good practice to ask occasionally that the answers be written not only in figures but also in words.

Numbers have not only magnitude, but order also. Order can be taught by allowing children to practise arranging number cards, within a limited range (say 70 to 90) in the right sequence. Or the children can be provided with incomplete sets of number cards and after arranging them in order are asked to write down the missing numbers, also in order.

Pupils show great interest in working through number combination cards, which not only teach and test them but also involve activity. On one side of these number combination cards is printed a number combination, whilst on the reverse side is printed the same combination but with the answer, e.g.: Of course, the cards must have some distinguishing mark (such as different colours or the top left-hand corners torn off) so that they can be stacked so that at first the answers are not visible to the children. The cards should be arranged in sets of increasing difficulty. The child takes a set, looks at the sum on the top card, makes a mental (or written) note of his answer and then looks on the reverse side of the card to see whether his answer was right. So he works through the sets. A spirit of playful competition can be introduced by letting pairs of pupils test each other.





8	14	9	12	6
13	7	20	10	16
16	1	15	7	14
12	8	10	18	20

A modification of the popular game of Lotto, alias Bingo, can be pressed into the service of teaching number facts. Several cards contain slightly differing sets of numbers printed on them (or the class can make the cards themselves). The card on p. 29 is an example of this.

The numbers are the answers to the addition (or subtraction, multiplication or division) combinations. The teacher calls out, say, 6+8. All children with the number 14 on their cards should cover the number with a piece of paper, or a bottle-top, or a button, etc. The child who first covers all the numbers on his card wins the game.

Cards of the kind shown below will be found of great use in practising addition.

	5	**6**	**7**	**8**	**9**
11	16	17			
12	17				
13					
14					
15					

The answers are, of course, written in the empty cells. These cards can be made quite easily from squared paper by the children themselves, who also fill in numbers corresponding to their mathematical attainment. Similar cards can, of course, be made for practice in the processes of subtraction and multiplication.

Such practical activities as reading the tape measure from both ends, e.g. 33+27, also provide valuable practice in addition.

Subtraction

THE TWO principal methods of subtraction* are those of *Equal Additions* and of *Decomposition.*

The Method of Decomposition

Eight units cannot be subtracted from 3 units, so "break down" one of the seven bundles of ten, leaving six bundles of ten in the "tens" column. One bundle of ten units added to 3 units makes thirteen units. Eight can now be subtracted from 13, leaving 5 units. Four bundles of ten subtracted from 6 bundles of ten leaves 2 bundles of ten.

$$\begin{array}{rr} \text{T} & \text{U} \\ 7^{6} & 3 \\ -4 & 8 \\ \hline 2 & 5 \\ \hline \end{array}$$

The Method of Equal Additions

This method is based on the principle that the difference between two numbers remains unchanged if the same number is added to each. Thus if two brothers are aged 7 and 3 years respectively, the difference in their ages will always be 4 years.

Eight units cannot be subtracted from 3 units, so give ten units to the top line, making it 13 units. We must also give one ten to the bottom line, making it 5 tens. Subtraction is now straightforward.

$$\begin{array}{rr} \text{T} & \text{U} \\ 7^{1} & 3 \\ 4^{5} & 8 \\ \hline 2 & 5 \\ \hline \end{array}$$

Much has been said about the relative advantages and disadvantages of the two methods. Some research workers have stated that the method of Equal Additions is so effective and so superior to that of Decomposition that the former should be generally adopted, and that the latter be abandoned. But W. A. Brownell carried out an experiment at the *learning* stage to compare the two methods when taught (a) in a mechanical way, and (b) in a rational way. His conclusions† were that:

---

*Instead of using technical terms such as *subtraction* and *division*, use the simpler words "taking away" and "sharing".

†Journal of Educational Research. Nov. 1947.

(i) It was impossible to decide between the two methods; it was the *teacher* that mattered.

(ii) If Decomposition is taught, it should be taught rationally.

(iii) If Decomposition is taught mechanically it compares badly with Equal Additions taught either way.

(iv) Equal Additions taught rationally does not enjoy a distinct advantage over the same method taught mechanically.

(v) In "transfer" tests, Decomposition taught rationally proved better since *understanding* functioned.

The advantage of the Decomposition method, which was brought to Europe from India hundreds of years ago, is that it is the easiest method to understand and the easiest to explain experimentally by methods such as the following.

A flat tray is divided into compartments marked Hundreds, Tens, Units. These contain bundles of beads (or sticks), the Units compartment containing single beads, the Tens containing bead-bars with ten beads each, whilst the Hundreds compartment contains ten bead-bars (100 beads). If the child has the problem of subtracting 37−19, he starts with 3 bead-bars of 10 beads each in the Tens compartment and 7 single beads in the Units. He will realise that he cannot take 9 from 7, so he breaks down one bundle of ten. He places these loose beads into the Units compartment which now has 17 beads. He now thinks of 37 as twenty and seventeen. Subtraction is now literally child's play. Nine from 17 leaves 8; 1 from 2 leaves 1. He is left with 1 ten and 8 units—18.

The method can be illustrated on the board thus:

	Tens	Units	T	U
37	3	7	2	17
−19	1	9	1	9
			1	8

The method of Equal Additions, though it is claimed to give greater accuracy and speed, has the disadvantage that it does not arise naturally out of situations which the child ordinarily meets. That is why attempts to teach it at too early an age meet with disappointment, the more so since it involves the artificial device that if equal numbers are added to two numbers their difference remains the same. The following is the method of Equal Additions:

To subtract, say, 157
from 342:

342	3 hundreds + 14 tens + 12 units
−157	2 hundreds + 6 tens + 7 units
	1 hundred + 8 tens + 5 units

Seven from 2 is not possible. Add 10 to the units in the top line and 1 (ten) to the tens in the second line. Seven from 12 leaves 5; put down 5 in the units answer.

Six from 4 not possible. Add 10 (tens) to the top and 1 (hundred) to the figure in the second line. Six from 14 leaves 8; put 8 in the tens answer.

Two from 3 leaves 1; put down 1 in the hundreds answer.

The method of Equal Additions may also be illustrated by using a tray divided into hundreds, tens and units.

If any children have already made a habit of either one of these two methods, it is very important that they should not be forced to change their habit.

As with addition, exercises should be graded and the following is such a grading:

(i)	57	89	35	48	77	96
	4	6	2	3	5	4
	—	—	—	—	—	—
	—	—	—	—	—	—

(ii)	68	579	384	895	737
	35	243	162	561	214
	—	—	—	—	—
	—	—	—	—	—

(iii)	38	658	974	749
	32	626	962	716
	—	—	—	—
	—	—	—	—

(iv)	32	74	615	532
	18	36	272	286
	—	—	—	—
	—	—	—	—

(v)	76	80	400	600
	30	28	127	204
	—	—	—	—
	—	—	—	—

Children should be taught to check their working by adding together the second and last lines, when the number so obtained should be that in the first line.

It is important that the teacher should inculcate into his children this very laudable habit of checking work.

Much practice in addition and subtraction will be given by the fascinating pastime of *magic squares*. The teacher will, at first, write a completed "square", on the board, e.g.:

8	3	4
1	5	9
6	7	2

Ask the class if they notice anything special about the figures. Don't tell the class that the figures in the rows, columns and diagonals total 15. Make the class tell you. Always encourage *the children* to think. Even when they have "discovered" for themselves that the rows and columns add up to the same total, make them think still further.

"What else is special about the figures?"

"Yes, when you add the figures cross-ways they again add up to the same total."

The class can then be asked to complete squares such as the following:

5		
3	6	
10		

4		8
	7	
		10

10	28	13
	17	

or

	9	
3	5	

Further magic squares can be made by:

(i) adding the *same* number to *each* of the numbers in the square;

(ii) subtracting each of the numbers in the square from any given number;

(iii) multiplying and (iv) dividing, each of the numbers by a given number.

Some of the abler children will not only be able to make up simple squares of their own with which to "test" their friends, but will also be able to tackle larger squares such as:

16		2	13
5	10	11	
9		7	
4			

A table of the calendar set out as follows can be used to give valuable practice in addition and subtraction.

1st Jan	1st Feb	1st Mar	1st Apr	1st May	1st June	1st July	1st Aug	1st Sep	1st Oct	1st Nov	1st Dec	1st Jan
0	31	59	90	120	151	181	212	243	273	304	334	365

Ask questions such as:

(i) How many days from 1st May to New Year's Day? (365 − 120)

(ii) How many days from New Year's Day to April Fools' Day? (90)

(iii) How many days until Christmas?

(iv) How many days left till your birthday?

Multiplication

As WITH ADDITION and subtraction, the complementary nature of multiplication and of division should be emphasised throughout. This will be done by always following questions such as "What are two fours?" by others such as "How many fours are there in eight?"

As a result of his earlier practice in counting in different intervals and studying the rhythmic pattern among numbers, the child will be familiar with sequences such as 2, 4, 6, 8, 10 . . .; 3, 6, 9, 12 . . .; 4, 8, 12, 16 . . . etc. Through such experiences he will build up the simple tables. P. B. Ballard asserted that to memorise the multiplication table without understanding it is worse—much worse—than understanding it without memorising it. The child will have realised through his own experimenting that when he has several equal groups to add together as 2+2+2+2, he need not count or add, since he will remember that four twos are eight. Similarly, other multiplication facts will have been encountered and unconsciously memorised in the course of his playful experiences. It is a very good practice to let the children fill in the facts they have discovered in a chart such as the following:

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12						
3	6	9	12	15	18						
4	8	12	16	20							
5	10	15	20								
6	12	18									
7	14										
8	16										
9	18										
10	20	30	40	50	60	70	80	90	100	110	120
11											
12	24										

Seeing the gaps in this table the child will be encouraged to fill in these gaps by his own efforts. Thus the gap 4×6 will be a challenge to an experiment with beads or peas or matchsticks. Exercises such as the following will help consolidate understanding of the multiplication and division tables:

Count out 24 matchsticks.

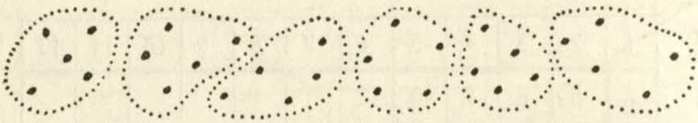
Divide the 24 into 2's. How many 2's are there?

Now divide the 24 into 3's. How many 3's are there?

Now into 4's. How many groups?

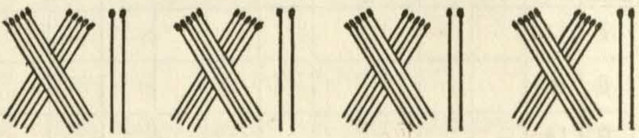
Now into 6's. How many are there?

A variation of this is to let the children make, say, 30 dots on their paper. They then have to encircle the dots in 2's; they repeat making the encircled groups contain 3, 5, etc., dots, as shown here:



The children then write down the number of groups they have made.

Matchsticks, etc., grouped as shown in bundles of tens and units can be very useful in illustrating multiplication processes. The following grouping illustrates 12×4:



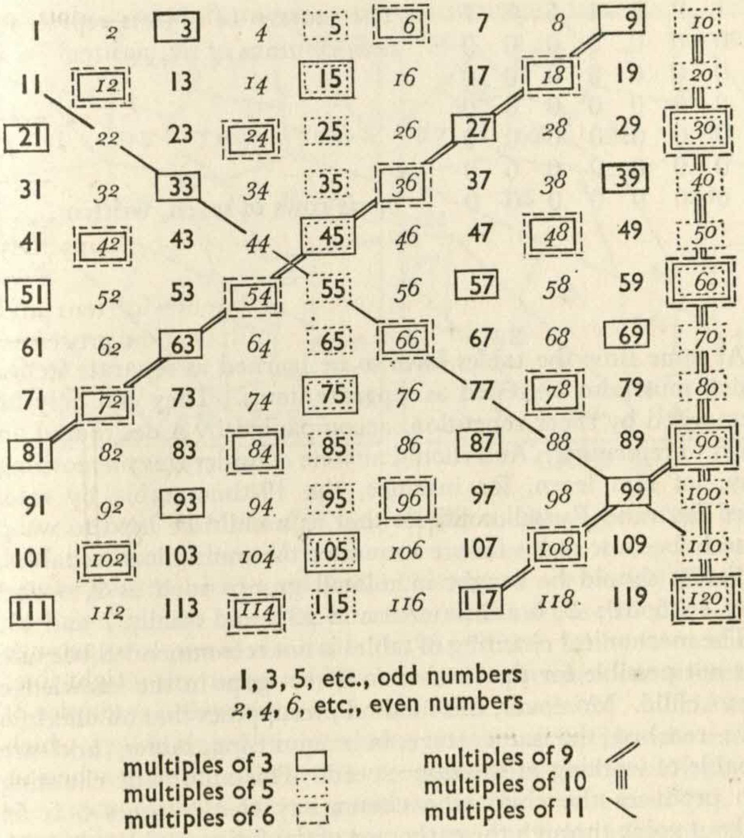
No attempt must be made to teach the child the tables formally by meaningless repetition.

It is very interesting and informative for children to arrange the numbers up to 120 as follows, and by means of straight lines drawn with coloured pencils to pick out the sets of multiples of the various numbers and to mark out tables of *factors* which divide into them. See the diagram on page 39. When teachers or pupils use this, they should represent the factors by colour.

The figures written alternately in ordinary and italic type show the series of odd and even numbers. The columns give intervals of 10. Numbers running diagonally downwards from left to right give intervals of 11; running downwards from right to left give intervals of 9.

Also distinguished on the table are the multiples of 3, 5, 6, 9, 10 and 11. The same idea may be worked out for the multiples of 2, 4, 7, etc.

Since at a later date it will be necessary to conserve the higher mental processes for solution of problems, it is essential that the tables be learnt and known so thoroughly that the results come automatically, almost as a reflex action. All subsequent work



will be vitiated unless the multiplication (and addition) tables are known thoroughly. A child whose mind is cumbered with undigested tables, with bonds which he has not himself bound, when asked what is 3 times 4, is just as likely to answer 7 as 12. The teacher should aim to strengthen in the mind the bond between, for example, 9 and 6 on one side and 54 on the other. Again, the fact that a child knows 6 times 7 as a fixed mechanical habit is no guarantee that he also knows 7 times 6 as a mechanical habit, and this relationship must be stressed.

By arranging objects in the following way the child can, of course, be made to "see" that 6×7 is the same as 7×6.

	1	2	3	4	5	6	7
1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0

The pattern of objects represents *seven* columns of six, written

$$\begin{array}{r} 6 \\ \times 7 \\ \hline 42 \\ \hline \end{array}$$

or *six* rows of seven, written

$$\begin{array}{r} 7 \\ \times 6 \\ \hline 42 \\ \hline \end{array}$$

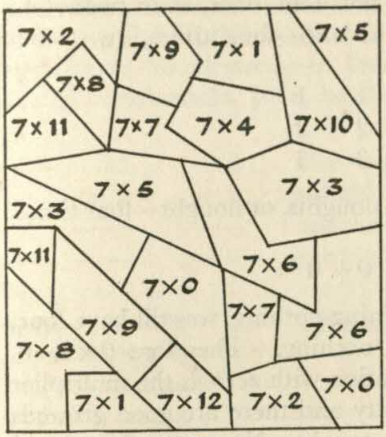
At some time the tables have to be learned as separate items, which must also be tested as separate items. They can only be memorised by sheer repetition, accompanied by a desire and an effort to remember. And should anyone consider this memorising easy, let him learn, for instance, the 19-times table by rote. Lord Bertrand Russell confesses that as a child he used to weep bitterly because of his failure to master the multiplication tables.

Tables should be taught in related groups such as 2, 4 and later 8; 5, 10; 3, 6 and later 9 and 12; and finally 7 and 11.

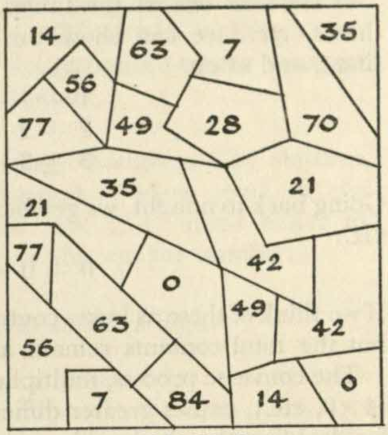
The mechanical chanting of tables is not recommended because it is not possible for the teacher to detect gaps in the knowledge of *each* child. Moreover, this method presupposes that all children have reached the same stage in memorising tables, and are capable of working at the same speed. This automatic chanting also produces the child who cannot say that 9 times 6 is 54 without going through the early part of the 6-times table. A much better practice is to let individual children recite the tables to each other in small groups.

Good practice in the tables can be given by removing one hand from a large clock-face, placing a figure, say 6, in the middle of the face, and as the remaining hand of the clock is moved to various figures on the clock-face, individual children are asked to give the answers to the multiplication of the two numbers in question.

Jig-saw cards can be an extremely valuable—and enjoyable—means of giving practice in mastering the multiplication tables. The teacher can easily make these jig-saws himself, using two



CARD A



CARD B

pieces of stiff cardboard A and B. Both these pieces are ruled into numerous irregularly shaped figures, care being taken to ensure that the shapes marked on each card are exactly the same, and also that no two shapes on any one card are identical. On the shapes of one of the cards (A) the 7-times table, for example, is written in any order whilst the correct products are written on the corresponding shapes on the other cards (B). This latter card (B), containing the products, is then cut up into the various figures. The child has then to place these irregularly shaped pieces on to their correct positions on the uncut card (A). He should, of course, take interest not so much in fitting the right *shapes*, but in learning or consolidating his knowledge of the tables. To ensure that the children solve the puzzle, not with the assistance of shapes, but through making use of their knowledge of tables, all outlines may be deleted from the base card (A), leaving only the numbers on it.

The table which needs most attention is the nought times (0×1, 0×2, etc.). This is the table which breaks down most frequently in practice, and yet many teachers consider it so easy that they never bother to teach it at all. To give the children experience of multiplication of nought by a number, i.e. the nought-times table, one could adopt the following method.

If we take one of the tables, say, four ones, four twos, four threes, etc., we can show this as one—four times, two—four times, and so on.

$$\begin{array}{c} 1 + 1 + 1 + 1 \\ 2 + 2 + 2 + 2 \\ 3 + 3 + 3 + 3 \end{array}$$

Going back to nought, we get four noughts, or nought—four times, viz.:

$$0 + 0 + 0 + 0$$

If we think of these as boxes containing nothing, we still have four, but the total contents remain as nothing. Therefore $0 \times 4 = 0$.

The converse process, multiplication with zero as the multiplier (4×0, etc.), causes greater difficulty and there are good grounds for deciding that it should not be taught. No matter how lucid the teacher's explanation, young children have not the maturity to understand the argument. Besides, the occasion to multiply by nought never arises. Even in cases like 24×20, etc., we do not multiply by nought. As explained on p. 44, multiplication by 20, 30, etc., should be treated as multiplication by 2 *tens*, 3 *tens*, etc.

When the multiplication tables have been thoroughly mastered, and when the child has appreciated their usefulness, he will be ready to tackle problems of written multiplication.

The following is a graded scheme of exercises to introduce the child to the progressive difficulties involved in multiplication.

(i)

6×9 or $\begin{array}{r} 6 \\ \times 9 \\ \hline \\ \hline \end{array}$

From his tables he will be able to answer questions of this type automatically. He will also appreciate that 6×9 is no different from 9×6.

Variety being the spice of life, these sums should later be set in this fashion:

$$\begin{array}{r} 9 \\ \times \quad \\ \hline 27 \\ \hline \end{array} \qquad \begin{array}{r} 6 \\ \times \quad \\ \hline 30 \\ \hline \end{array} \qquad \begin{array}{r} 8 \\ \times \quad \\ \hline 32 \\ \hline \end{array} \text{ etc.}$$

Working such sums will be a useful preparation for division. Of course, should the children have difficulty in tackling these, they should be allowed—in fact, encouraged—to use counters such as match-sticks, peas, bottle-tops, etc.

(ii)	$\begin{array}{r} 23 \\ \times\ 3 \\ \hline \\ \hline \end{array}$	$\begin{array}{r} 43 \\ \times\ 2 \\ \hline \\ \hline \end{array}$	No "carrying" involved. The children should themselves have come to the conclusion that it is much easier to multiply by the smaller number.
(iii)		$\begin{array}{r} 42 \\ \times\ 4 \\ \hline \\ \hline \end{array}$	Bringing in an extra digit.
(iv)		$\begin{array}{r} 36 \\ \times\ 4 \\ \hline \\ \hline \end{array}$	Involving "carrying", but keeping in the same decade.
(v)		$\begin{array}{r} 76 \\ \times\ 8 \\ \hline \\ \hline \end{array}$	"Carrying" and going into the next decade.
(vi)	$\begin{array}{r} 40 \\ \times\ 2 \\ \hline \\ \hline \end{array}$	$\begin{array}{r} 302 \\ \times\ 3 \\ \hline \\ \hline \end{array}$	Involving noughts.
(vii)	$\begin{array}{r} 20 \\ \times\ 4 \\ \hline \\ \hline \end{array}$	$\begin{array}{r} 24 \\ \times 20 \\ \hline \\ \hline \end{array}$	Preliminary work with 20, 30, etc., as multipliers will offer a good introduction to multiplication by two digits. Here it needs to be stressed that since we are counting in tens, no units figure will occur, and hence 0 must be placed in the units column.

The very common formula "add a nought" for multiplying by 10 is deprecated. 36×10, for example, means the value of 36 tens, and this is not brought out by blindly "adding a nought". Multiplication by 20, 30, etc., should be regarded as multiplication by 2 tens, 3 tens, etc.

(viii)

$$\begin{array}{r} 24 \\ \times 23 \\ \hline \\ \hline \end{array}$$

Multiplication should begin with the left-hand digit of the multiplier. This has the advantage that the first line of the product gives a rough approximation to the final answer.

This procedure also proves advantageous when, at a later stage, decimals are multiplied, or when contracted methods and approximations are required.

$$\begin{array}{r} 36 \\ \times 21 \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 78 \\ \times 41 \\ \hline \\ \hline \end{array}$$

The teacher should also include questions such as these. This is necessary because with single digit multipliers the operation of multiplying by one is never performed, thus making multiplication by numbers such as 31, 51, etc., confusing to children at first.

$$\begin{array}{r} 362 \\ \times 7 \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 234 \\ \times 7 \\ \hline \\ \hline \end{array}$$

Multiplication sums can provide useful practice in tables for weaker children. Thus, giving a child who is weak, in the 7-times table, for example, numerous sums of this kind, will give him so much familiarity with the table that he will soon memorise it, not only as a saving of time and effort, but also as a defence against drudgery.

Division

NUMEROUS EXPERIENCES such as knowing that since there are 4 legs to a chair, 5 chairs will have 20 legs and conversely, that if 20 legs can be seen, they belong to 5 chairs, will have given the child his first notions of the related nature of multiplication and division. He will have learnt that since 12 pennies make a shilling, 24 pennies mean 2 shillings. Through "sharing", through finding how many times one magnitude is contained in another, he will have consolidated his notions of division. Before actually tackling sums in division, the teacher must ensure that the child has had adequate experience of situations involving division. On a question such as sharing 20 marbles among 3 boys, the pupil will have discovered that each boy receives six marbles, and that two marbles are left over. Drawing simple diagrams will also be of use in introducing him to the notion of division. The notion of division can also be brought home to young children by giving them cards, each costing, say, 1s. or 20 cents, but which have different numbers of buttons on them. The child will soon discover that on a card with two buttons, each button is worth 6d. or 10 cents; on a four-button card, 3d. or 5 cents, etc.

In the course of learning the multiplication tables the child will also have built up the division tables as the reverse aspect of the multiplication tables. This is one reason why division should be taught concurrently with multiplication.

A scheme of division sums becoming increasingly difficult would be as follows:

(i) $6\,\overline{)\,30}$,* etc.

(ii) Involving no remainder and no carrying: $3\,\overline{)\,96}$, $4\,\overline{)\,848}$

(iii) With a remainder, but no carrying: $2\,\overline{)\,45}$, $3\,\overline{)\,638}$

(iv) With no remainder, but involving carrying: $4\,\overline{)\,76}$

(v) Involving a remainder and carrying: $3\,\overline{)\,76}$

(vi) Involving placing the quotient correctly: $6\,\overline{)\,126}$

(vii) Involving noughts: $4\,\overline{)\,416}$, $3\,\overline{)\,960}$

Again, a reminder is necessary that children make many errors when division involves noughts.

---

*Children should be taught to set out their work in this form, rather than $6\,\underline{|\,30}$, since there will then be no need to make any change when the class begins long division.

All the above are, of course, examples of *short division*. Children should be encouraged to use short division when dividing by numbers less than 12, or by "easy" numbers such as 100, 200, 120, etc.

Of course, at first the more difficult sums should be done by the *long division* method, e.g.:

$$\begin{array}{r|r} & 63 \\ \hline 9 & 567 \\ & 54 \\ \cline{2-2} & 27 \\ & 27 \end{array}$$

Later, the children will be able to do much of the routine working in their heads, when they will set out their working thus:

$$\begin{array}{r|r} & 63 \\ \hline 9 & 567 \end{array}$$

For divisors greater than 12, *long division* will be used. Many teachers expect ten-year-olds to master long division by two figures. However, research has shown that mastery of this process requires a mental age of twelve. Topics must not be taught until the children are mentally able to understand them.

(viii) $21 \overline{)\,84}$

Children find difficulty in determining the quotient figure. They can be helped in this by doing preliminary work on division using multiples of 10. The children soon realise that division by a number such as 20 reduces itself to division by 2. Then, employing divisors which are very near multiples of ten, e.g. 21, 31, etc., the children will consider the result of dividing by, say, 21 as being near to that obtained when dividing by 20, and they will accordingly estimate the quotient by using the divisor 2.

This process prevents the children from blind and aimless guessing of the quotient figures. The early examples should be so chosen that estimates obtained in this way are the correct ones. Division by numbers such as 19, 29, etc., comes later.

(ix) Involving a remainder: $31\,\overline{)\,36}$
(x) Where the quotient is not so readily apparent:
$24\,\overline{)\,56}$ $43\,\overline{)\,98}$
(xi) Placing the quotient: $32\,\overline{)\,768}$
(xii) With a remainder: $23\,\overline{)\,489}$
(xiii) Involving nought: $34\,\overline{)\,704}$
(xiv) Using divisors with figures greater than four: e.g. 39, 58, etc.

In sums like $12\,\overline{)\,439}$ children often write the answer as 36+7 remainder. They must be made aware that this is not strictly accurate, since the 36 is 36 *groups* of twelve and the 7 is a remainder of seven *units*. The answer should be 36, remainder 7.

Another common error is for children (and teachers) to say "four divided into twelve" for 12÷4. They should not, for this statement may be confused with that of "four divided into twelve parts", which is quite a different matter. The correct statement is "twelve divided by four".

Division should be checked by reversing the process, viz., multiplying the quotient by the divisor and adding the remainder. It would, however, be wasteful of time to check every division by multiplication. Children should be encouraged to make partial checks; but this requires more intelligence than is generally available. Moreover, whilst a check applied intelligently and deliberately is a valuable help, to the unintelligent and lazy a check often becomes merely an additional source of error and confusion. It must also be borne in mind that the working of a check should not be more complicated than the original working.

The doing of "sums" in the four fundamental processes can be very much enlivened by giving the sums in such a way that they contain missing numbers which have to be filled in, e.g.:

$$\begin{array}{r}36\\+{*}3\\\hline 79\\\hline\end{array}\qquad\begin{array}{r}2{*}\\+{*}4\\\hline 81\\\hline\end{array}\qquad\begin{array}{r}87\\-3{*}\\\hline {*}4\\\hline\end{array}\qquad\begin{array}{r}47\\-{*}{*}\\\hline 25\\\hline\end{array}\qquad\begin{array}{r}{*}\\\times 9\\\hline 63\\\hline\end{array}$$

$$\begin{array}{r}{*}7\\\times 5\\\hline 8{*}\\\hline\end{array}\qquad\begin{array}{r}4{*}\\\times 6\\\hline 2{*}4\\\hline\end{array}\qquad\begin{array}{r}8\\7\,\overline{)\,{*}6}\end{array}\qquad\begin{array}{r}{*}6\\6\,\overline{)\,27{*}}\end{array}\ \text{etc.}$$

Children will love such problems which appeal to their "puzzle" instinct.

Drill

ANY SUBJECT matter is likely to be forgotten, no matter how well it has been taught or how well it has been mastered, unless it is maintained by repeated application and practice. If the mastery of a process is to be a real possession, the memory needs tickling at intervals. Unless there is some drill the lessons are written on sand. Moreover, in mathematics, many operations need to be performed not only correctly but with facility and speed and some even automatically. For example, if a child doesn't master tables when young he is hampered in all his subsequent work and wastes more time than their accurate memorisation would have required. This can only be secured through repeated and systematic practice, i.e. through drill. It has been well said that practice without the power of mathematical thinking leads nowhere; that the power of mathematical thinking without practice is like knowing what to do but not having the skills or tools to do it.

Cards will be found very useful in giving practice, not only in addition, but also in subtraction, multiplication, and division. Considerable interest will be added if the cards are played as a game. One player takes a pack of cards, each with a simple sum, e.g. 24÷6, written on it.

He puts down one such card on the table. His opponent, whose pack of cards contains only numbers, has to put down the card with the appropriate answer, e.g. 4. Each correct answer wins a point. The packs are changed round after a while, so that both players get practice. As a variation, one player puts down a card with a number on it, e.g. 54; the other player then has to put down the 9×6 card (assuming they are practising multiplication).

Cards containing one or several problems are very valuable, not only for work in addition, but also in subtraction, multiplication and division. For the less able child they are more satisfactory than text-books: the unit of work is small, the objective is easily seen, and they give more opportunities to the weaker child to sense that desirable feeling of exhilaration when he beams, "I have got *all* my sums right."

Many inveigh against drill as being futile. The old pedagogy undoubtedly laid too much stress on it. But criticism of mechanical drill can go too far. In games we still insist that practice makes perfect. The alleged drudgery exists more for the teacher than for the child. In reasonable quantities mechanical work is enjoyed by children for it satisfies their "rhythmic" instinct, it satisfies their need for success, and it provides satisfaction from the increase of speed which results from the increasing mastery of the operation.

Whilst unimaginative drill of a formal, mechanical and meaningless kind cannot be too severely deprecated, it must not be forgotten that young children, satisfying their urge for repetitive action, love testing their number facts by doing lots of "little sums", the more so when they find they are getting them right. Short periods of ten or fifteen minutes devoted to drill will be found invaluable. Because the attention span of children is not great, longer periods of drill will become tiresome and hence ineffective. Keep the drill periods short, regular and bright.

Drill should be subsequent to, and not a substitute for, understanding of meaning. Children should not be drilled on procedures which they do not first understand. Under such circumstances drill lacks both significance and motive. As Dr. F. J. Schonell so rightly points out: "practice without understanding pays poor dividends, but also understanding without plenty of practice yields similarly reduced results."

Facility without understanding is valueless. Without understanding and motive, drill becomes mere drudgery. It should be related to some purpose which is significant to the pupil. Among devices that may be used for motivating drill work are games and competitions between teams, improvement charts, etc.

Whenever possible, drill should be so conducted that the pupils can work at their own differing rates and according to their different abilities.

Of course, drill must not necessarily be confined to written work. *Viva voce* drill or "mental" work can be most effective. Not only does it take less time than written work, but it provides the teacher with a fine opportunity of discovering the child's method.

Projects and Activities

PRIMARY SCHOOL teachers must ever beware of taking written work too far and of making the lessons formal. Now that the children have to some extent mastered the written language of arithmetic and the four fundamental processes of addition, subtraction, multiplication and division, this is the jumping-off point for children to make their play activities and experiences even more meaningful mathematical acquisitions. Through study of broad environmental topics and through investigations of situations and phenomena at first hand, they will develop their mathematical ideas still further. Activities such as the following all offer valuable mathematical opportunities:

(i) Shopping—the grocer's shop—market.
(ii) Transport—fares—seating accommodation—times.
(iii) Post Office—postage rates. The Bank.
(iv) The farm—quantities of food—costs, etc.
(v) Measuring. Weighing.
(vi) Models—making of a church, etc.—idea of scale—maps.
(vii) Games—marking out a games pitch—marking goal circles (using a rope and stick to draw the circles)—marking out squares for hop-scotch.
(viii) Opening out cardboard containers of various shapes, examining their outline and the various lines of folding and cutting. Constructing containers from the shapes.
(ix) Plotting routes using simple compass directions.
(x) Time—calendars—diaries.

The task of the teacher is to provide an abundance of such opportunities, himself or herself merely acting as a guide, whilst the children learn from their own experiments and discoveries. Instead of children sitting as silent prisoners at their desks, the class should consist of small groups or individuals intent on a variety of affairs. Movement and speech should be quiet and purposeful; whilst the children should have come to know that the interferer and the slacker are not going to be tolerated. The teacher must be on the look-out for signs of readiness for new fields to conquer, when by approval, suggestion and the provision of new material, he or she will stimulate their advance. Slogging away at the same activity day after day will certainly breed boredom and lack of interest, in the wake of which always follows indiscipline. The children's interest in a particular activity is a very good guide to its usefulness and appropriateness at a particular time. The successful teacher is he or she who knows when and how to arouse curiosity, when and how to provide not only the material and apparatus for children's experimentation but also the mathematical language through which their ideas and concepts will become more explicit. It is a good idea to have a weekly rota of shopmen and assistants, farmers, bankers, postmen, travel agents, etc., so that all children can enjoy the highlights of actual experience. Dressing up for the various parts will add considerably to the enjoyment of the numerous activities.

Shopping activities are not only very attractive to young children but also provide a very fruitful experience of number and money knowledge. The class "shop" should be stocked with all kinds of commodities that appeal to the children. In order to allow many different activities to go on at the same time it is often a good idea to have several "stalls" handling commodities of differing kinds. Of course, all objects for "sale" should be priced. Whilst the class will generally use cardboard or paper money, real money should be used from time to time. Strips can be used for 10s. and £1 notes (or dollar notes).

When a child buys something from the class "shop" for 4d. (or 4 cents), and another article for 3d. (or 3 cents), the separate prices are lost in the total 7d. (or 7 cents). But if the shopkeeper gives him 5d. change from 1s. then this 5d., representing money still to be spent, will help consolidate the 7, 5, 12 relationship. Similarly, 3 cents change from 10 cents, consolidates the 7, 3, 10 relationship.

The Basic Processes with Money

Compound Addition

In countries where decimal currency, such as dollars and cents, is used, addition—and subtraction, multiplication and division—are quite straightforward, e.g.:

$$\begin{array}{r} \$\ \text{c.} \\ 3{\cdot}75 \\ 4{\cdot}50 \\ \hline 8{\cdot}25 \end{array}$$

That is why so many countries have changed to decimal notation.

Adding shillings and pence, and later £ s. d.* offers no great problems either. Just as the method of working through "ten" is to be deprecated in ordinary numerical addition, so too should the method of working through the shilling (or hour or any other quantity) be avoided. Children should be taught to add the total number of pence, e.g. "eight, fourteen, seventeen", etc., and then to convert to shillings. By this means their concentration is not distracted. Moreover, the method of making a total and then converting to a higher unit is of more general application and more economical of mental strain. And, of course, when he encounters such problems as 1s. 9d. × 7 he will first obtain 63 as the number of pence which he will later convert to shillings and pence.

---

*In the third century B.C. the Romans first used a silver penny, which was called a *denarius*. Hence our "d" for penny. £ (or l) comes from *libra* which refers to the balance used for weighing gold pieces. Our shilling sign "s" is from *solidus*, a gold coin used in Rome.

The first exercises set in compound addition should involve no carrying, e.g.:

s.	d.
4	2
3	4
6	3

——

—— etc.

Later there should be carrying, e.g.:

s.	d.
7	6
9	11
2	3

——

——

STATE INSTITUTE OF EDUCATION ★ BANIPUR ★

Compound Subtraction

To avoid confusion, the same method of subtraction (Decomposition or Equal Addition) must here be used as in ordinary subtraction with numbers. The process may be illustrated, if necessary, by setting out the amounts in coins in boxes or trays divided into compartments for pounds, shillings and pence. Apart from this, there are no intrinsic difficulties involved in subtraction with money or other measures. But the teacher must always beware of assuming that because a topic seems ludicrously easy to an adult mind that that same topic holds no difficulty to the young growing mind. The following incident, recorded by Miss L. D. Adams in her book *A Background to Primary School Mathematics*, p. 96, illustrates how a simple, straightforward problem can become most confusing to immature minds. The problem was:

"My slippers cost me 9s. 6d. and my jumper 2s. 6d. less. How much was the jumper?"

Rosie read the sum, and Billy, who sat beside her, worked it out orally as 7s. for the jumper.

Rosie: "Oh yes, let's put that down" (neither knew how to show the work).
Eventually they wrote down:

$$\begin{array}{rr} \text{s.} & \text{d.} \\ 9 & 6 \\ 7 & 0 \\ \hline \\ \hline \end{array}$$

Rosie: "Is it add up, take away, or multiply?"

Billy: "Let's look back at yesterday's page."
They looked back in Billy's book and found that a similarly set sum in which he had subtracted had been marked wrong, so they decided to "add up"!

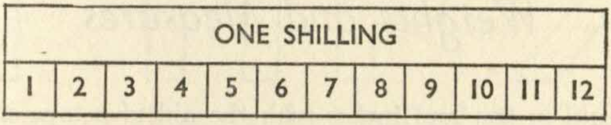
Billy: "I've got the answer, Rosie—15s. 6d."
Rosie looked at Billy's book and without hesitation copied down his answer.

Compound Multiplication

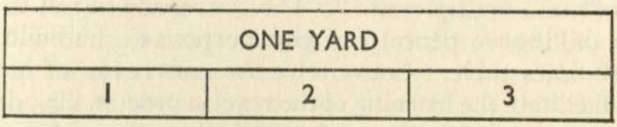
In the Primary School, children will rarely need formal multiplication of money, and what exercises are set should be limited to very simple ones, the multipliers being less than 12.

Before closing the topic of money as a measure, it must be pointed out that such a topic presents difficulties which other systems do not possess. These difficulties arise from the fact that whereas twelve inches do *make* a foot, and a foot can be divided into twelve equal parts called inches, twelve pennies do not *make* a shilling—they are equivalent to it; nor can a shilling be divided up into twelve equal parts.

The following strip-method will be found invaluable in illustrating the various equivalents in money, weights and measures, etc. The children should, of course, make the strips themselves.



12 PENNIES=1 Shilling



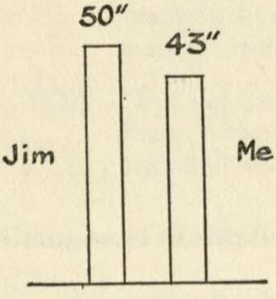
3 FEET=1 Yard

Weights and Measures

Length

Measuring in feet and inches with the aid of a tape measure not only provides valuable mathematical experience but will also help to build up a sound knowledge of the composition of 12 and of small multiples of 12. The conversion of feet to inches (as with shillings to pence) can give purpose to the building up of the 12-times table. Conversely, the conversion of inches to feet will facilitate the learning of the reverse process, viz., division. If the measurements be in yards and inches, the tables of three will be considerably practised.

The children's estimates of their journey to school or other fairly short distances in terms of their strides can be used to link up with the making of plans and maps.

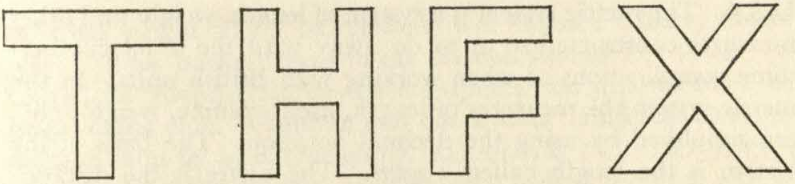


Questions such as "How much taller than you is Jim" will be the occasion for the child and for Jim to mark each others' heights, measure them, and then to record the difference. Making a simple drawing as that shown will be of assistance to the child's comprehension.

A task such as "Draw the blackboard in your book" will give children considerable practice in measuring and will also introduce them to the notion of *scale* drawings.

Another exercise that will be found enjoyable and which will also give the children comprehension of scale, is to give them cards on which letters or figures and, later, even simple drawings are printed. The class have to draw these but making their drawings twice, four times, etc., as big.

Another interesting activity is to give children pictures on a square network. The children have to draw the picture twice, thrice, etc., as large. Or the picture can be drawn by the teacher



on a square network on the blackboard, and the children have to copy the picture into their books. Many children will be engrossed for hours making their own pictures on a square network. An amusing variation on this activity is to draw caricatures by enlarging one dimension, say the head, out of proportion to the other dimensions.

Having gained the notion of scale, some of the more intelligent children will be able to find distances on maps by measuring the length of the line drawn between places or the length of a piece of cotton which traces a particular journey marked on the map.

The historical development of units of measure will always prove very interesting to young children. The smaller units of length were suggested, understandably, by the various parts of the body. The *inch* was at first the length of the thumb joint, but later became the length of three grains of barley. The width of twelve thumbnails formed a *foot*, and King Henry I decreed that the *yard* was to be the distance from the point of his nose to the end of his thumb.

The "span" or spread of the hand, and the "palm", or width across an open hand at the base of the fingers, were common units. The "cubit" was the length of the forearm from the elbow to the tip of the middle finger; the "digit" the breadth of a finger; the "rod" was the "length of the left feet of 16 men lined up as they left church on Sunday morning". The *fathom* was the length across two arms outstretched. The *mile* was originally, but no longer, the distance covered by a man walking 1,000 (mille) paces.

By the eighteenth century a confusing assortment of units had grown up. After the French Revolution in 1789 the opportunity was seized to establish a more uniform system. This was the *metric* system which was adopted in France in 1799 and is now used in nearly every civilised country except Britain and the

U.S.A. The metric system is a system of length, weight and other measures constructed so as to do away with the usual cumbersome complications as when working with British units. In the metric system the measures of length, area, volume, weight, etc., are simplified by using the decimal notation. The basis of the system is the length called a *metre*. The metre is the distance between two scratch marks on a bar of platinum-iridium when the bar is at the temperature of melting ice (0° C.). This distance is slightly more than a yard in length—39·37 in.—and is one ten-millionth of the quarter-circle from the North Pole to the Equator.

The children will probably want to know why a bar of platinum-iridium is used, or why temperature has to be considered. This will provide a good opportunity of linking mathematics with elementary physics.

Weight

THE CHILDREN'S "shopping" and other activities should also result in acts of weighing and of measuring. The class should make and use their own price lists and ready reckoners for goods sold at the "shop" by weight, etc.:

1 oz. costs 3d. (or 3 cents)
2 oz. cost 6d. (or 6 cents) and so on.

Children are fascinated when given the opportunity to weigh things either with spring balances or kitchen balances. They will enjoy finding their own weight and the weights of various common objects. The teacher should play upon this fascination children have for weighing. Weighing letters and parcels in the class "Post Office" and then calculating the postage necessary, will afford valuable mathematical opportunities. Weighing objects together to find their total weight, and finding differences in weight, will illustrate addition and subtraction. Weighing together several (say 6) similar objects of equal weight, e.g. 2 oz. packets, and checking by weighing that six times 2 oz. is 12 oz. will give practice in multiplication. Finding how many objects of equal weight (say 2 oz. packets) will balance a heavy object (say a 12 oz. packet) leads to the division sum of $12 \div 2 = 6$. Of course, for young children the multipliers and divisors must be single digits only. Children can check their multiplication and division sums by actual weighing. Comparing children's weights to find who is the lightest, heaviest, etc., to find the total weight of teams, will give useful practice in converting stones to pounds and vice-versa.

No formal work in calculation with weights and measures should be undertaken unless and until the children themselves have handled and become familiar with the measures and have discovered for themselves the relation between any pair of successive units. The sums, restricted to simple calculations, should arise out of practical tasks. Moreover, no problems should be set involving an impracticable range of units such as miles—yards—feet—inches, or ton—cwt.—qr.—lb.—oz. Artificial problems such as reducing millions of inches to miles are equally useless. Written reckoning with money, weights and measures should be limited to reckoning for which the children can see a clear purpose. Their problems will generally be limited to pairs of successive units such as lb., oz.; s., d.; dollars, cents; ft., in., etc.

Capacity

YOUNG CHILDREN delight in experimenting with sand or water to compare the capacities of various containers such as milk or other bottles, pint cans, watering cans, mugs, cups, petrol tins, etc. They should get to know the "feel" and size of common quantities such as the gallon, quart and pint. A project on the milkman's deliveries to the school, the cost of milk per pint, and so on, can be made to furnish valuable practice in computation.

Time

THE CONCEPT of time is a difficult one for young children, and so the subject should not be introduced too early nor should it be hurried when it is. A large clock-face with movable hands (or an old clock) will be invaluable in teaching the general ideas of telling the time, of a.m. and p.m., the 24-hour day and so on. A simple historical approach to the measurement of time will appeal to children. They will love making experiments on time using such measures as a sand "egg-timer", a candle into which pins have been stuck, a pendulum (consisting of a weight attached to a piece of string about 1 metre in length) which beats seconds, or by recording the movement of shadows. The attempt to estimate seconds with fair precision will appeal as a game to be played. Experiments on distance travelled in given times and on the times required to go certain distances will give appreciation to what is meant by "fast" and "slow".

Once again, long exercises involving formal calculations of days, hours, minutes and seconds should never be set. Of course, there is no objection in letting the child solve his little problems involving conversions of hours to minutes or minutes to seconds and vice-versa.

Oral and Mental Arithmetic

Whilst the children will now be doing written work, oral arithmetic must not be neglected since it makes great contributions to the arithmetic lessons. Not only does it vary the routine of the arithmetic lesson; quick animated questions directed to all corners of the class arrest the immediate attention of the class. Such questions will also help to foster speed in performing simple calculations. Oral arithmetic can and should be a means of quick revision which will give the teacher some idea of the progress of individuals. Finally, of course, by means of simple oral exercises and examples the teacher is enabled to introduce new topics to his class.

Short Cuts

The teacher should not lose the opportunity of bringing short cuts to the notice of her children whenever the occasion arises. Thus if the children are engrossed in a problem involving the addition of a number such as 99, she should point out—assuming they have not discovered the short-cut for themselves—that much time and effort can be saved by mentally adding 100 and then subtracting 1. Similarly, if 198 has to be added to a number, since $198=200-2$, it is far easier and quicker to add 200 to the number and then subtract 2.

In subtracting, say, 98 from a number, since $98=100-2$, first subtract 100 and then add 2. Similarly, in subtracting, say 204 from a number, since $204=200+4$, first subtract 200 mentally and then subtract 4.

If a number has to be multiplied by 50, it is easier to multiply by 100 mentally and then divide by 2.

Similarly, since $25=\frac{100}{4}$, multiplication by 25 is simplified by multiplying mentally by 100 and then dividing by 4. Likewise, to multiply by 250, multiply mentally by 1,000 then divide by 4.

$125=\frac{1}{8}$ of 1,000; so to multiply by 125, multiply by 1,000, then divide by 8.

If a number has to be multiplied by 4 and then by 5, one multiplication by 20 will give the answer more speedily.

The cost of 5 articles at 2s. 11d. each, is 5d. less than 15s.

Numerous short cuts are possible where decimal currency is used.

Finding the cost of 10, 100 or 1,000 articles is no great problem, e.g. cost of 100 articles at 16c. each $=\$0{\cdot}16\times100=\$16$.

The cost of 5 articles = $\frac{1}{2}$ the cost of 10 articles.
The cost of 15 articles = cost of $10+\frac{1}{2}$ cost of 10.
The cost of 50 articles = $\frac{1}{2}$ cost of 100.
The cost of 150 articles = cost of $100+\frac{1}{2}$ cost of 100.
The cost of 25 articles = $\frac{1}{4}$ cost of 100 articles.
The cost of 98 articles = cost of 100 articles − cost of 2 articles.
The cost of 26 articles = $\frac{1}{4}$ cost of 100 articles + cost of 1 article.

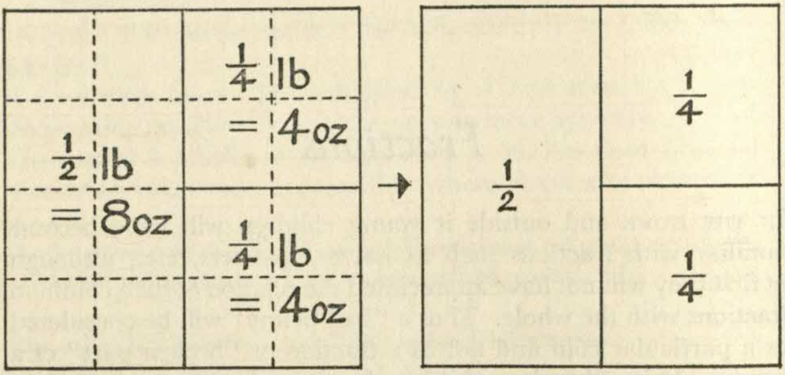
Fractions

IN THE HOME and outside it young children will have become familiar with fractions such as halves, quarters, etc., although at first they will not have appreciated the relation of such common fractions with the whole. Thus a "half-penny" will be considered as a particular coin and not as a fraction or "broken part" of a penny. It is only when children have reached the age of eight or nine that they begin to appreciate the fraction concept; and this they do more or less incidentally.

In the course of the child's experiments with weighing, measuring, etc., he will come to conceive the notion of a fraction. Thus the child who discovers in play when filling and emptying bottles of water, that four bottlefuls always fill a quart measure and that the quart can be redistributed to fill four bottles, will realise the relationship of the four parts to the whole. Fractions should be taught informally as and when they arise naturally in the child's experience. Gradually the child will realise that there can be half a page, half a number, half a pound, etc., and that these fractional or "broken" parts exist without the necessity of having the whole from which they were broken. A cake divided fairly into, say, six parts will introduce the concept of "sixths" and so on. Children find it hard to realise that the bigger the denominator the smaller the piece. They will soon get the idea, though, if you ask them whether they would rather have $\frac{1}{5}$ of a cake, or $\frac{1}{7}$!

From their experiences with money, with measuring, etc., they will have come to appreciate that 1d. is $\frac{1}{12}$ of a shilling, that a foot is $\frac{1}{3}$ of a yard and so on.

Diagrams such as the following form a useful intermediate step between the pupil's concrete experiences and the final abstract idea of a fraction—half, quarter, etc., which have no separate real existence but are ideas arising from experience.



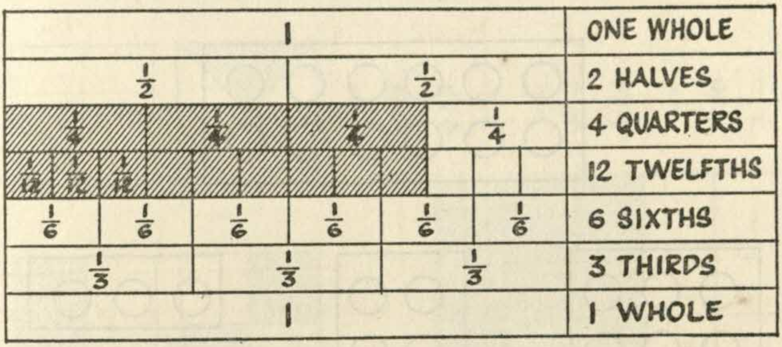
Some idea of what young children can appreciate of the concept of fraction at different ages and stages of maturation is given below. Most children of seven are aware that there are two quarters in a half, six eighths in three quarters; and that $2\frac{1}{2}$d. must be added to $9\frac{1}{2}$d. to make twelve pennies or one shilling. By the age of eight they will have extended their ideas to realising that six halves make three wholes; that eight quarters make two wholes; that there are 15 minutes in $\frac{1}{4}$ hr., that 90 minutes is $1\frac{1}{2}$ hours, that 1 day is $\frac{1}{7}$ of a week, etc. By the age of nine they will have come to appreciate that two sixths make $\frac{1}{3}$, that one third is larger than one quarter, that $\frac{3}{12}=\frac{1}{4}$, that there are four $1\frac{1}{2}$d. in 6d., that $\frac{5}{12}$ of a foot is 5 inches and so on.

Folding strips of paper into halves, quarters, thirds, sixths, etc., and placing them side by side, as below, will give children valuable insight into the nature of fractions or "broken parts" and of wholes, and also of the equivalence of some of these fractions:

$$\text{e.g. } \frac{1}{2}=\frac{2}{4}=\frac{3}{6}=\frac{4}{8}=\frac{6}{12}$$

The children can further verify such equivalence by drawing a line $\frac{1}{2}$ in. long and then, using different fractional parts, they can see that this length is also $\frac{2}{4}$ in., and so on.

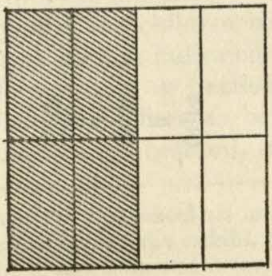
This equivalence will be a valuable introduction to the notion of the *common denominator*, a concept of vital importance for future progress with fractions.



The shaded portions of the strips show that $\frac{9}{12}=\frac{3}{4}$.

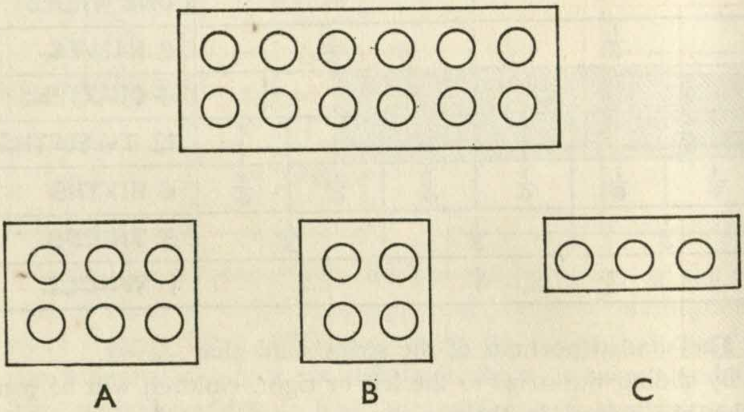
By sliding the strips to the left or right, children will be considerably helped in their early attempts at addition and subtraction of fractions.

It is important that children be aware that they are dealing with the relationship of a part to the whole. This idea will be obscured if the children cut up, for instance, a square into two halves, when two new wholes are produced. This difficulty can be prevented by starting with two identical squares, cutting up only one of them, and then fitting the parts on to the uncut square. By this device children can learn that, for instance, four eighths will cover half a square, i.e. $\frac{4}{8}=\frac{1}{2}$; that two eighths will cover a quarter of the square, i.e. $\frac{2}{8}=\frac{1}{4}$.



$$\frac{1}{2}=\frac{2}{4}=\frac{4}{8}$$

Another method of teaching equivalence is to give the children cards with buttons on them. They must find out how many buttons are covered by card A, card B, card C, etc.

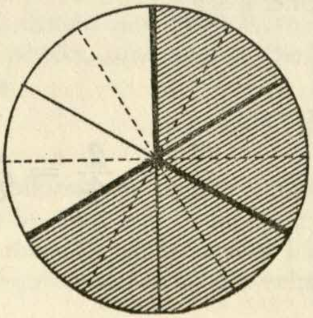


Card A, which is half the size of the original card, will cover 6 of the 12 buttons, thus giving appreciation of the fact that $\frac{1}{2}=\frac{6}{12}$. Similarly, B will show that $\frac{4}{12}=\frac{1}{3}$; whilst C will show that $\frac{3}{12}=\frac{1}{4}$.

The children will love trying to solve problems such as the following by folding or cutting or drawing.

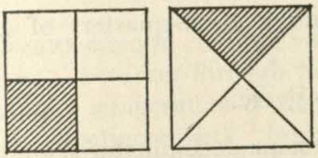
"Share 3 (or 5, etc.) squares among four people."

Cutting sectors of a circle does not suffer from the disadvantage of obscuring the relationship of a part to the whole.

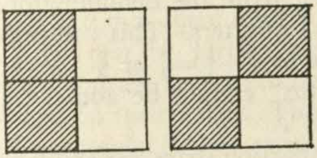


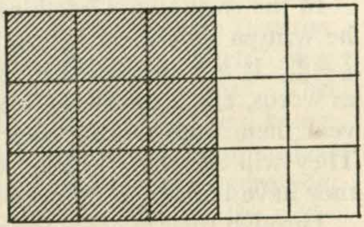
$$\frac{2}{3}=\frac{4}{6}=\frac{8}{12}$$

The following figures show further ways of obtaining fractions by folding, cutting, covering squares or rectangles.



$$\frac{1}{4}+\frac{3}{4}=\frac{4}{4}=1 \quad 1-\frac{1}{4}=\frac{3}{4} \quad 1-\frac{3}{4}=\frac{1}{4}$$





$$\frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2} \qquad \frac{3}{5}=\frac{9}{15} \quad 1-\frac{3}{5}=\frac{2}{5}$$

At this point the teacher could lead up to the rule for converting fractions to a common denominator.

By referring to the above equivalent fractions, viz. $\frac{2}{3}=\frac{4}{6}=\frac{8}{12}$ and by appropriate questioning, the children *themselves* should formulate the rule that a fraction is not altered if its "top" and "bottom" are multiplied by the same number. The pupils should then be asked to convert simple fractions using this rule they have discovered. They can check their results by folding strips, by measurement or by some other practical means with which they are familiar.

Only after the children have had very considerable practical experience such as indicated above should they begin to attempt more formal work in fractions.*

At first all fractions to be added or subtracted will have the same *denominator* (bottom), e.g. $\frac{2}{4}+\frac{1}{4}$; $\frac{2}{6}+\frac{3}{6}$; $\frac{4}{5}-\frac{3}{5}$, etc.

---

*In the Primary school no useful purpose will be secured by extending fractions beyond the eighths, tenths and twelfths families. On occasion, perhaps, sixteenths and twenty-fourths may be tried.

Next will come fractions whose denominators, though different, all belong to the same family, as:

$$\tfrac{3}{8}+\tfrac{1}{4};\ \tfrac{3}{4}-\tfrac{1}{2}$$

(which can be well illustrated by using three-quarters of a shilling and half a shilling;

$$\tfrac{2}{3}+\tfrac{3}{6};\ \tfrac{4}{5}-\tfrac{1}{10};\ 1-\tfrac{1}{3},\ \text{etc.}$$

In the early stages fractions with the same denominator should be written separately before they are added or subtracted, e.g. $\frac{7}{8}-\frac{3}{8}$. It is also helpful if the children write the denominators as words, e.g. 2 quarters+1 quarter=3 quarters. This will prevent them from writing such nonsense as $\frac{1}{4}+\frac{1}{2}=\frac{2}{6}$ or $\frac{2}{3}+\frac{3}{6}=\frac{5}{9}$. They will see that "thirds" and "sixths" cannot be added till they have both been reduced to "sixths".

Through their experiments with comparison strips the children will have acquired a notion of common denominator, and will now be able to tackle more difficult problems of addition and subtraction of fractions where both denominators have to be changed, e.g. $\frac{1}{2}+\frac{1}{3}$; $\frac{1}{4}+\frac{2}{3}$; $\frac{11}{5}+\frac{1}{2}$; $\frac{2}{3}-\frac{2}{5}$, etc. Children, in working these sums, will probably formulate the rule that the common denominator is obtained by multiplying the denominators. That this is not always necessary can be brought home to the children by problems such as $\frac{3}{4}+\frac{1}{6}$; $\frac{5}{8}-\frac{1}{6}$.

Multiplication and division of fractions should be left to the secondary school. Rather than forcing manipulative skill, give the children abundant opportunities to acquire variety and depth of practical experience.

Decimal Fractions

PRIMARY SCHOOL children will have become acquainted with the decimal notation through hearing adults speak of a time of 9·7 sec., of a temperature of 98·4°, of a rainfall of 0·4 in., etc. From their experience with "broken parts" of a whole when experimenting with fractions, the children will come to realise that the figure after the decimal point similarly represents "parts" of a whole and that there are ten such "parts", called "tenths". The children can be further familiarised with "tenths" by being made to draw and measure in inches and in tenths of inches marked on their rulers. This work can be made more interesting and purposeful if they record the growth of a plant or the rainfall, etc., to the nearest tenth of an inch. The convention of writing tenths in the decimal notation, e.g. $\frac{4}{10}$ as 0·4, can now be introduced.

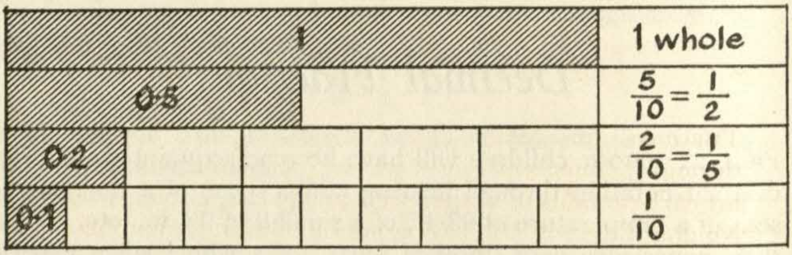
One way to show the need for a decimal point is to write some numbers, including numbers with tenths, in columns on the board, e.g.:

645·6, 234·5, etc.

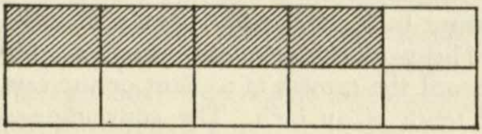
Hundreds	Tens	Units	Tenths
6	4	5	6
2	3	4	5
3	9	2	7
5	2	3	4

When the lines forming the columns are rubbed out, leaving only the numbers, it should become obvious to the class that a method of differentiating the units figure is necessary—that is, the decimal point.

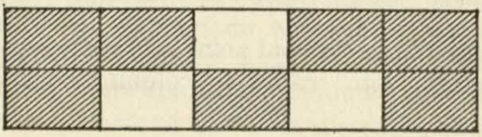
A diagram such as the following will be found very useful in illustrating the meaning of decimal parts.



Familiarity with the decimal notation will be given by making children answer what decimal fraction of the whole the shaded (or unshaded) portions of cards such as the following represent:

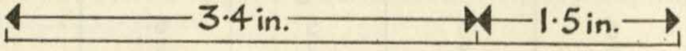


$\frac{4}{10}$ or 0·4 shaded
$\frac{6}{10}$ or 0·6 unshaded

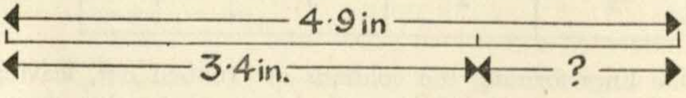


$\frac{7}{10}$ or 0·7 shaded
$\frac{3}{10}$ or 0·3 unshaded

Simple addition and subtraction of decimals provide no difficulties. They can be illustrated by drawing and measuring lines, e.g.:



$$\begin{array}{r} 3{\cdot}4 \\ +\,1{\cdot}5 \\ \hline 4{\cdot}9 \end{array}$$



$$\begin{array}{r} 4{\cdot}9 \\ -\,3{\cdot}4 \\ \hline 1{\cdot}5 \end{array}$$

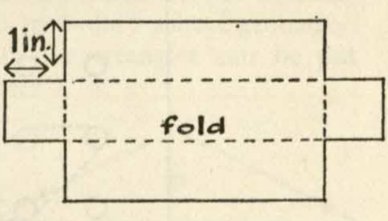
Multiplication and division of decimals must be left for the Secondary School, though many Primary children will be able to multiply simple decimals by simple whole numbers, using not some formal rule for computation of decimals (which belongs to the Secondary school stage) but plain common sense.

The Beginnings of Geometry

THROUGH HIS GAMES and toys, like hoops, hop-scotch, etc., through playing with jig-saws, etc., the child will have received his early notions of shapes such as squares, circles, triangles, etc. No doubt, he will also have a notion of what is meant by "streamlined"! Later he will become conscious of solids such as cubes and cylinders.

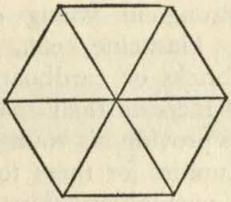
The constructional instinct is strong in young children. Through modelling in sand, clay, Plasticine, etc., through building houses and garages from bricks or cardboard or by cutting and folding paper, they will increase their spatial perception. The task of the teacher is to provide his young charges with sufficient apparatus and opportunities for them to enlarge their spatial concepts through their own observations, experiments and constructions. Such apparatus will include cubes, pyramids, prisms, flat tiles, cylinders, discs, spheres, hemispheres, semi-circles, etc. The children should be left to experiment and to make their own discoveries about shapes. A child will very soon discover that a rectangle rolled to make a cylinder will make a suitable funnel for his model engine.

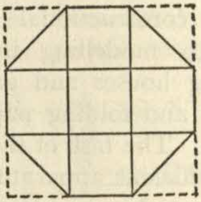
Another interesting piece of construction work is the making of a paper tray with sides, say, 1 in. high. Let the children experiment and work this out by their own efforts, through their own trials and errors. When the child sees that a plan, such as the following, when cut out and folded as indicated will yield the desired tray, he will have made a great advance in his ability to perceive spatially.



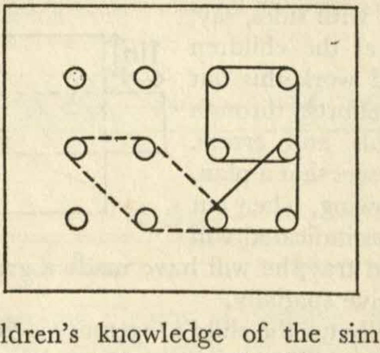
What will really tax the child's ingenuity is the making of a tent from paper. Don't tell him what to do or how to do it. He will learn immeasurably more from the experience of doing it himself by trial and error.

Actual drawing of the various figures will help to consolidate concepts of shape. Of course, since young children find it difficult to handle compasses, they will draw circles either by drawing round a coin or by using a pencil attached to the free end of a fixed piece of string (the pin and loop method). In the course of his experimental drawings he will come to see that a hexagon is formed from six equal-sided triangles; that the octagon can be made up from squares and half-squares (i.e. triangles) and so on.



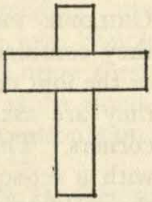


The following homemade piece of apparatus can be used effectively in teaching children about two-dimensional shapes. The board contains movable pegs. String is twisted round three or more pegs to produce various geometrical figures which the class are asked to identify. Turning the board round will give the children the valuable experience of the effect of rotation.

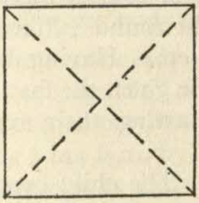


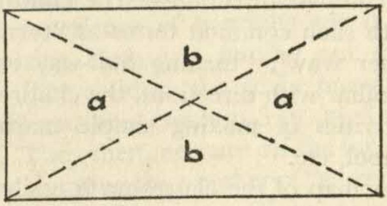
In testing children's knowledge of the simple geometrical shapes the teacher must beware of asking questions such as "What is a square?" The young geometer will be able to answer such questions satisfactorily only when he is in the secondary school even though he *knows* the shape of a square. The question asked should rather be in the form "Which of these is a square?"

First ideas of symmetry can be seen when the young child places two pieces of wood as shown to represent the wings and body of an aeroplane. Playing the game of folding over a piece of paper containing a wet signature to produce a "ghost", likewise results in a symmetrical pattern (or "smudge" as the children will call it). So also will pricking through a pattern on folded paper, or cutting curves in a doubled paper, when, on opening out the paper, a symmetrical mirror image is obtained.

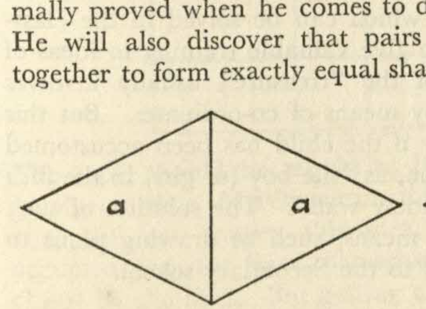


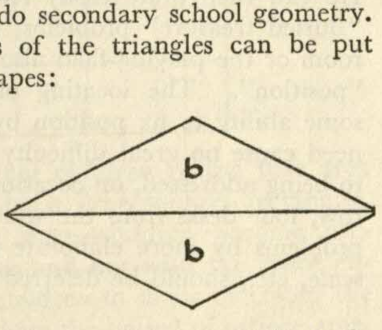
Through experiments with folding in the diagonals of a square or of a rectangle, cutting out the triangles so formed, and then





manipulating them in a variety of ways, the child will make quite a few discoveries about diagonals, which he will find formally proved when he comes to do secondary school geometry. He will also discover that pairs of the triangles can be put together to form exactly equal shapes:





For those schools and those children that can afford the luxury of Meccano sets, a large amount of spatial and geometrical experience is there for the tapping.

Angles

CHILDREN PROBABLY become aware of the idea of an *angle* when they consider the shape of corners, whether they be of a room, or the four corners of a page. This concept is strengthened when they are asked to find whether a small square will fit in the corners. This testing for "square corners" can be done either with a set-square or, better still, with square corners made by the children themselves by folding a sheet of paper twice. These ready-made set-squares can then be put to use in drawing right-angles, and even half right-angles by folding. This could be a suitable jumping-off point for dealing with the important topic of compass directions. The children will, of course, be familiar with such common terms as "turning right round", "facing the other way", "turning half-way round", etc. Having become familiar with directions, the children can be given the fascinating activities of making simple maps, of charting their route to school, etc.

A map of the classroom is easily made. The child counts the number of paces required to cross the room in each direction. He then counts the same number of "squares" on his paper. He can then quite easily locate the place where he sits, etc. "Buried treasure" problems, which can be solved in the classroom or the playing-field also give valuable training in ideas of "position". The locating of the "treasure" usually involves some ability to fix position by means of co-ordinates. But this need cause no great difficulty if the child has been accustomed to being addressed, on occasion, as "the boy (or girl) in the fifth row, four desks from the window wall". The solution of such problems by more elaborate means, such as drawing plans to scale, etc., should be deferred to the Secondary school.

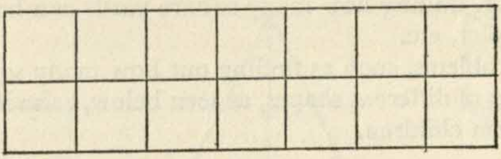
INSTITUTE OF EDUCATION MANIPUR

Area

THIS IS ANOTHER difficult new concept for young children. In spite of that, some misguided teachers plunge straight into teaching formulae such as Area = Length × Breadth, etc. Such children who have had a quantitative stage forced upon them before they have had personal practical experience of the qualitative stage, will think of area as merely a multiplication sum: they will have no idea that a *surface* is involved.

Such children will quite happily multiply inches by feet to give square inches. And if they find their sum is marked wrong they will change their answer to square feet—that *must* be right!

It is quite artificial and valueless to launch into areas unless the class have had adequate experience of handling pieces of wood or cardboard, of measuring them and finding out how many "squares big" they are. The children should be provided with pieces of cardboard of different sizes (such as 3 in. by 3 in., 2 in. by 4 in., 6 in. by 2, etc.). They then measure off the inches, rule the pieces as shown, and by counting the number of "squares" or covering them with inch-square tiles they will be able to determine how many "squares big" the respective pieces are.

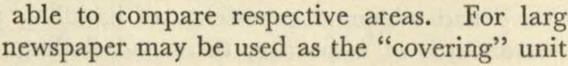


The topic should be taught so that children realise that area involves surface. Merely getting them to draw squares, rectangles, etc., will not convey this idea, since children become preoccupied with the lines themselves and not with the space enclosed by the lines. But getting children to shade or colour the space between the lines will give them the notion of surface area. Painting squares on a pattern is another useful and enjoyable activity.

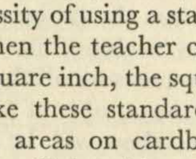
Of course, merely getting children to draw, say, a rectangle and then to rule off and count the squares contained in it, is a purposeless task. The children will probably soon show you that they think so, too. The teacher must make these area activities interesting, by clothing them in the guise of purposeful "problems" that the young "discoverers" must solve. For instance, finding the number of milk bottles contained in a crate, finding how many square cartons or bricks will form one layer of a rectangular box, how many tiles will cover a certain area of the floor, how many trees each needing a square of certain area, can be planted in an orchard, etc. Children will also gather valuable notions of area from their modelling and constructional work, from their work in laying out "plots" and "beds" in the school garden, etc.

By finding out how many similar exercise books, etc., cover the tops of their desks and how many are needed to cover a table, they are able to compare respective areas. For larger areas sheets of newspaper may be used as the "covering" unit. Later, the necessity of using a standard unit for comparison will probably arise, when the teacher can give a very elementary introduction to the square inch, the square foot and the square yard. Children can make these standard comparing units by cutting out the required areas on cardboard or paper. With these units the children will be able to perform valuable exercises in square measure; e.g. finding how many square yards can be fitted into the garden plot, etc.

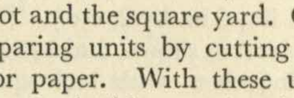
Simple problems, such as finding out how many squares there are in figures of different shapes, as seen below, raise illuminating reactions from children.



12 squares



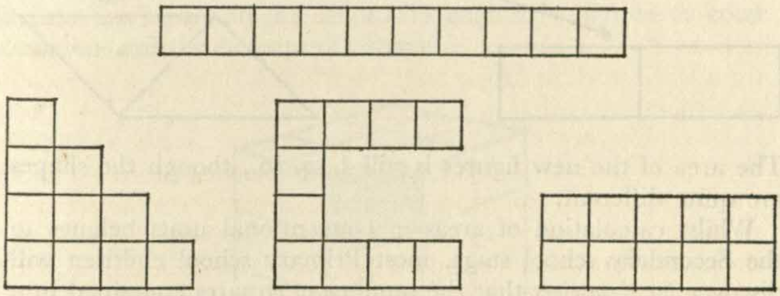
12 squares



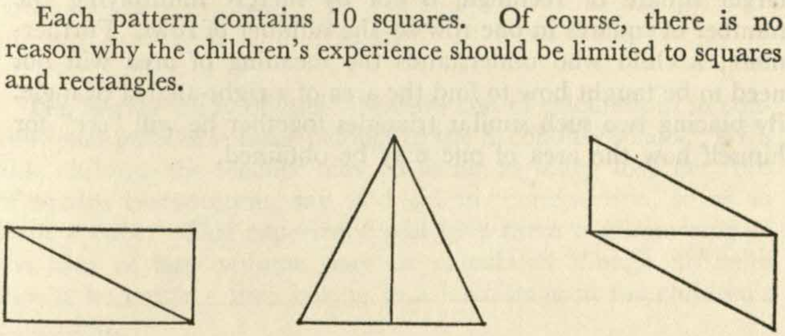
12 squares

It comes as quite a shock to discover that different shapes can have equal areas. This is a concept the child only becomes aware of after actual experiment, just as he realises only from his experiments that differently shaped vessels may hold the same amounts of liquid.

By arranging tiles or square pieces of paper or cardboard into different patterns the children are made further aware of this difficult notion, namely that the same number of tiles *cover* the same space though they are arranged quite differently.

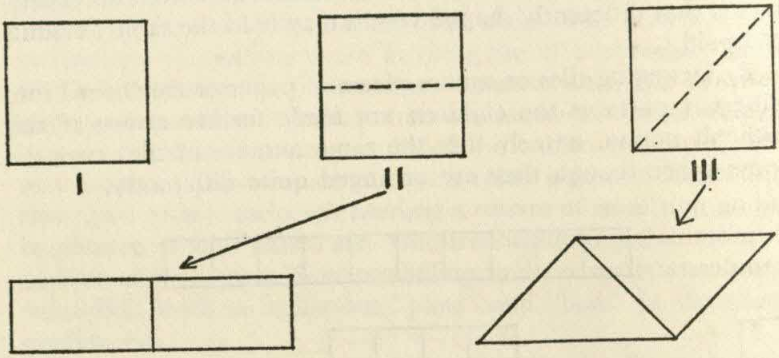


Each pattern contains 10 squares. Of course, there is no reason why the children's experience should be limited to squares and rectangles.



Each pattern contains two triangles of the same area. By reproducing these and others patterns on squared paper, the children will have a valuable record of the equal areas they have "discovered".

Another interesting activity is to make the children draw three 1-in. squares. The children are asked to cut the second and third squares as indicated and to join the cut pieces in such a way as to form a figure other than a square, e.g.:

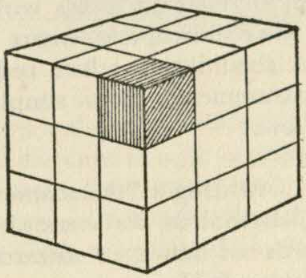


The area of the new figures is still 1 sq. in., though the shapes are quite different.

Whilst calculation of areas in conventional units belongs to the Secondary school stage, most Primary school children will discover *for themselves* that the number of squares contained in a larger square or rectangle is got by merely multiplying the number of squares in one row by the number of rows. Furthermore, a child who understands the meaning of area will not need to be taught how to find the area of a right-angled triangle. By placing two such similar triangles together he will "see" for himself how the area of one may be obtained.

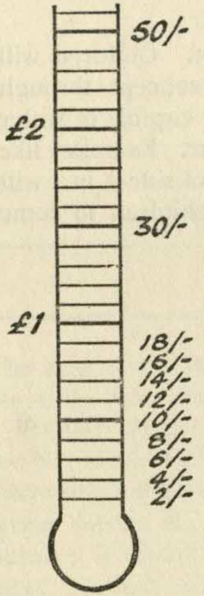
Volume

LIKE AREA, volume is another difficult concept. Children will have received their first glimmerings of this concept through their "experiments" such as finding how many cupfuls of water or spadefuls of sand, etc., fill a certain container. Exercises like finding out "how many little blocks (i.e. cubes of side 1 in.) will fit into this box," are all helpful in enabling children to come to understand the concept of volume.



The further study of cubic measure and of calculations in conventional units of volume belongs to the Secondary school. With able children the teacher may go as far as piling together rods of square cross-section, say of 1 sq. in. cross-section, so as to form a cube. This experience will give them a glimmering of the idea of how volume may be calculated though no rules should be taught—they belong to a later stage in the children's education.

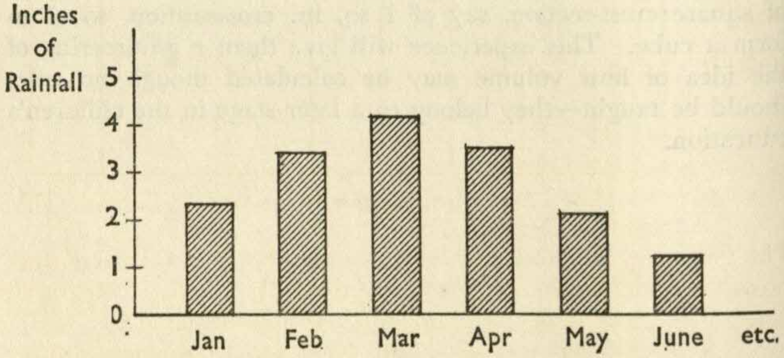
Graphs



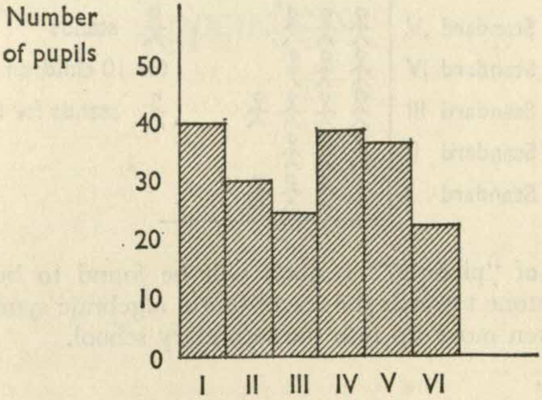
WHILST THE CONCEPT of one length standing for another length, as in scale drawing, is difficult for young children, the idea of a length or area to represent some other quantity, as in graphs, is even more difficult. Even so, Primary school children will not only be able to tackle the simplest kinds of graph, such as *histograms* (column charts), but will also relish drawing and filling them in. Of course, the graphs should relate to topics within the experience of the children, and, where possible, the children should themselves read and record the measurements. Some simple suggestions are as follows:

1. Making a "thermometer" kind of graph which records, for example, class savings towards an outing or towards a donation to a Children's Home, etc.

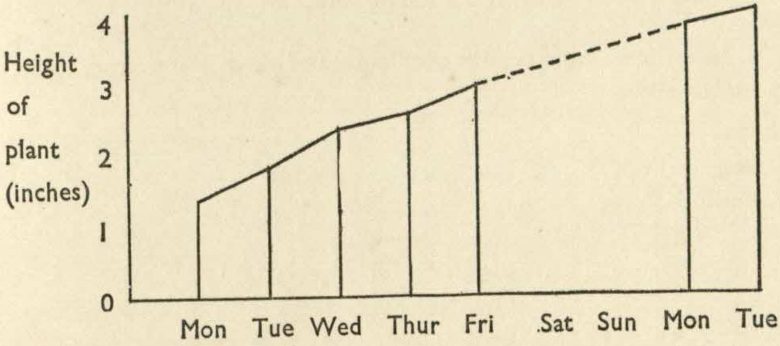
2. Another simple type is the "column" record of temperature, of rainfall, etc.



3. Again, the children could make a graphical record of the number of children in the various classes in their school, e.g.:



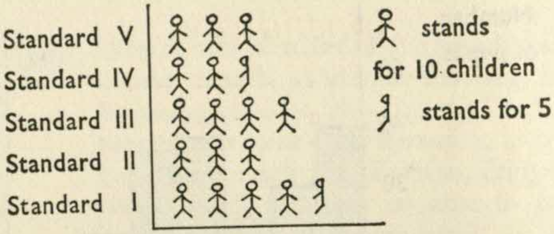
4. They could keep a daily or weekly record of the height of, say, a bean plant. The plant in a jar is placed on a shelf behind which is a sheet of paper fixed to the wall. The plant grows a fixed distance daily and its corresponding height is recorded on the paper, giving a result something like this.



The gap resulting from the school being closed at the week-end provides the opportunity for asking the class whether we could "guess" the height of the plant on these two days.

5. The children will also find great fun in making simple

isotypes, in which a pictorial symbol represents a group of units, e.g.:



The use of "pictorial" symbols will be found to be a useful stepping-stone towards the use of literal algebraic symbols when the children move up into the secondary school.

Appendices

Appendix A

An Account of Current Experiments in the Teaching of Elementary Mathematics in the United States of America.

Exciting and, in some ways, revolutionary experiments into mathematics teaching at the elementary level are now being conducted in the U.S.A. These experiments were born out of the increasing dissatisfaction with both the content of elementary mathematics and the methods of teaching it.

The "traditional" mathematics curriculum consists largely of learning *facts* and *techniques*. Instead of being based on childrens' *natural modes of learning*, the teaching consists mainly of "telling", "showing", "drilling" and, of course, "criticising". The new mathematics programmes all stress ability to think creatively about mathematics. The concepts, and the structure and logical development underlying the principles of mathematics are emphasised, without sacrificing any development of skills.

The theory of *sets* is of importance in all branches of modern mathematics. It is not surprising, therefore, to find that nearly all the new programmes use the concept of sets to give a more accurate introduction to the four fundamental processes of arithmetic.

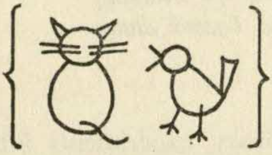
Professor Patrick Suppes, of Stanford University, California, the Director of the *Stanford Project*,* holds that all mathematics can be developed from the notion of sets and operations on sets. Even very young children have the idea of a set as a collection or family of things. This knowledge is utilised to provide a simple and concrete introduction to number operations which is both easy to explain and to understand. A great advantage is that arithmetical operations such as addition and subtraction may be introduced in the very early stages without confusing the issue with Arabic numerals.

---

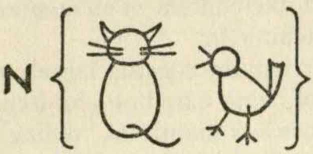
*Further particulars about the Stanford Project may be obtained from: The Director, Stanford Project, Ventura Hall, Stanford University, Stanford, California, U.S.A.

The progression from objects to number is made as follows:

(i) The first abstraction the children make is the description of a set. This description is made by picturing the members of the set and enclosing the pictures in brackets. For example, the set consisting of a cat and a bird would be represented as shown.



(ii) The next step in abstraction is the jump to the property of *number*. The symbol N is used to indicate that the *number* of objects in a set is being considered and not the objects themselves; that is, we are concerned not with what the members of the set are but with *how many* members there are.

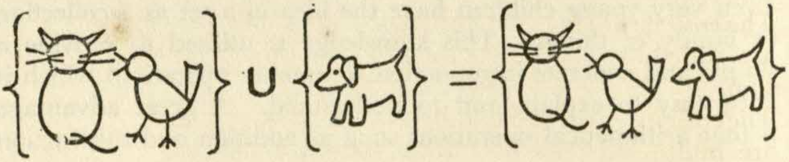


This notation names the number ("two", in this example) but at the same time it maintains the pictorial representation of the set. This device of abstracting from the particularity of the objects in a set to the property of number serves as a "transition" to:

(iii) The final stage when Arabic numerals are used e.g. 2 in the case illustrated above.

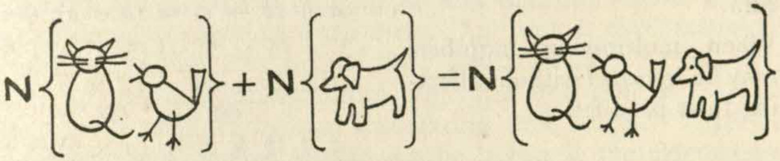
The following development illustrates the progression towards abstraction in the case of addition.

(i) The first stage is the concrete operation of putting sets together, i.e. of forming their union.



(The symbol "U" indicates the union of the set containing the cat and the bird with the set containing the dog giving one set containing all three animals.)

(ii) In the next stage the N notation is used to indicate the addition of the *numbers* characteristic of each set, without considering the particular objects themselves.



A set of two objects and a set of one object are equal to a set of three objects.

(iii) In the final stage Arabic numerals are used:

$$2 + 1 = 3$$

The number zero, which usually causes trouble, presents no difficulty when it is considered merely as the empty set $\{\quad\}$, the set which has no members.

Experiments have shown that children introduced to number and number operations using this "sets" approach proved markedly superior to a control group of children in writing numerals and in arithmetical operations.

The Mathematical Basis of Operations with Numbers

The following simple examples illustrate how the principles involved in the commutative, the associative and the distributive laws* of mathematics are utilised in the new experimental programmes so as to give children a greater awareness of the "why" and the "what" and the "how" of mathematics.

The commutative law

The *order* in which two numbers are added does not affect their sum.	$4 + 5 = 5 + 4$ $a + b = b + a$
The *order* in which two numbers are multiplied does not affect their product.	$3 \times 6 = 6 \times 3$ $a \times b = b \times a$

---

* Of course, no mention would be made of the words "commutative" "associative", etc., to elementary school-children.

The associative law

When adding, numbers may be *regrouped* without affecting their sum.

$$\begin{aligned} 2 + (4 + 7) &= (2 + 4) + 7 \\ \text{i.e. } 2 + 11 &= 6 + 7 \\ a + (b + c) &= (a + b) + c \end{aligned}$$

When multiplying, numbers may be *regrouped* without affecting their product.

$$\begin{aligned} 3 \times (4 \times 5) &= (3 \times 4) \times 5 \\ \text{i.e. } 3 \times 20 &= 12 \times 5 \\ 60 &= 60 \\ a \times (b \times c) &= (a \times b) \times c \end{aligned}$$

It is this associative property that we use in addition and subtraction:

$$\text{(i)} \quad 29 + 8 = (29 + 1) + 7 \text{ or } 30 + 7$$

$$\begin{array}{lrl} \text{(ii)} & 36 & = 30 + 6 \\ & +49 & = 40 + 9 \\ \cline{2-3} & & 70 + 15 \text{ or } 80 + 5 \text{ or } 85 \end{array}$$

$$\begin{array}{lrl} \text{(iii)} & 53 & = \ 40 + 13 \\ & -28 & = -20 + 8 \\ \cline{2-3} & & 20 + 5 \text{ or } 25 \end{array}$$

The distributive law

This combines addition and multiplication.

$$\begin{aligned} 3\,(5 + 6) &= (3 \times 5) + (3 \times 6) \\ 3 \times 11 &= 15 + 18 \\ a\,(b + c) &= (a \times b) + (a \times c) \end{aligned}$$

This is the property used in multiplication, e.g.

$$\begin{array}{lrcrcr} \text{(i)} & 38 & & 30 & & 8 \\ & \times\ 7 & & \times\ 7 & & \times\ 7 \\ \cline{2-2}\cline{4-4}\cline{6-6} & & = & 210 & + & 56 \end{array}$$

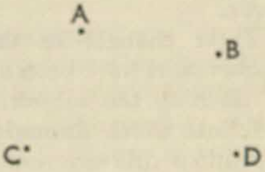
$$\begin{array}{lr} \text{(ii)} & 46 \\ & \times 35 \\ \cline{2-2} \end{array}$$

$$= (5 \times 6) + (5 \times 40) + (30 \times 6) + (30 \times 40)$$

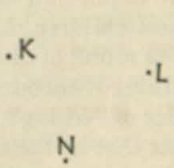
The traditional elementary mathematics programmes have concentrated mainly on the facts of the four fundamental processes. Evidence has accumulated to show that children can learn fairly complex concepts and that they can begin learning them much earlier than had been considered possible. Young children are capable of learning *far more* than the regular school programme attempts to teach them. And so it is that instead of restricting themselves to the four fundamental processes the new programmes include simple algebraic and geometric ideas to provide challenging and stimulating experiences. There is no reason why algebra should not be begun in the elementary school. The ideas of algebra are often so much easier and involve a great deal less intellectual effort than the operations of numerical calculations. For example: "Find a number which will make the sentence $4 + \square = 13$ true" involves simpler thought processes than those for "Divide 79 inches into 12 equal pieces".

These exercises, based on the Stanford Project, illustrate the sort of simple geometry that can be successfully taught in elementary schools.

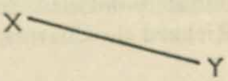
(i) Draw AB.
Then draw AC, CD and BD.
Then join C and B.
How many triangles do you see?



(ii) Draw the circles which have KL as radius.
In how many points do these circles intersect?
Label them M and N.
Draw KM, and then LM
Use arcs with radius equal to KL to find whether
(a) KL=KM (b) KL=LM (c) KM=LM
What kind of triangle is KLM?



(iii) Construct an equilateral triangle with base XY.



Other geometric ideas introduced are:
angles; line segments; arcs; construction and bisection of angles; construction of perpendiculars; squares; etc.

The use of appropriate vocabulary in the elementary school always poses a problem. The new programmes avoid the use of extensive vocabulary since that generally obscures the concepts in the case of young children. Their aim is to get the children to think about basic concepts, instead of parroting technical terms which have no clear meaning for them. The vocabulary can come after the child has thought about the concepts in a creative way for some time.

Inter-planetary and sub-atomic distances are commonplace in the language of today's society. The new programmes place greater emphasis on helping children to understand and appreciate very large and very small numbers. Likewise, in keeping with the mathematical demands of present-day society, greater emphasis is given to creative thinking, to estimating in round figures, and less to the purely computational aspects of arithmetic.

These changes in the traditional content of elementary mathematics have been accompanied by changes in the methods of teaching the subject.

Efforts which formerly went into finding ways to motivate repetitious drill are now directed to seeking ways of presenting challenging situations, within the capabilities of the children, which will encourage them to make their own discoveries and so give depth and breadth to their mathematical insight. Since, for most children, life up to the age of ten or eleven years is a glorious round of discovery and exploration, a discovery approach to learning is encouraged. Hughes Mearns blames the traditional practice of "telling", "showing", "drilling" and "criticising" for the fact that by the time children reach the end of the elementary school, any enthusiasm they might have had for mathematics has long since been knocked out of them. Omar Khyyam More, of Yale University, has remarked that if you want your daughter to stop playing with dolls, all you need do is give her careful, minutely-detailed instructions on "how to play with dolls". Richard de Charms, the psychologist, has added ". . . and also adhere to a rigid time schedule for playing with dolls, reward her for playing with dolls, and periodically evaluate her progress in playing with dolls."*

Children must regard their teachers not as persons standing over them waiting to pass judgment on them, but as partners in the business of joyous intellectual discovery. Accordingly, classes are conducted with a maximum of student participation, with the children thinking and talking and doing instead of just listening and accepting. Children are encouraged to be re-creators of mathematics, not just learners of mathematics. Such methods, of course, imply a freedom to make mistakes and to question.

If a student makes a mistake, the teacher would allow him time to discover his mistake for himself and to profit from his discoveries. Concepts of mathematics explored together can be equally fascinating to both teacher and taught and the teacher needs to be able to judge when the time is ripe for further developments of the topic under discussion or for branching out into other fields.

Many teachers will say that there is nothing "new" in what has been written above. That is true. Good teachers have been using these methods all along; they have always insisted on reasoning instead of rote memorisation. The purpose of the new programmes is to help all teachers do better what many teachers have been doing well for a long time; to enable all children to profit from the enlightened methods which some children have been benefiting from all along. It is not their intention to replace entirely the regular present-day programmes. In fact, most of the programmes claim that their work is within the framework of existing curricula. They suggest new ways of organising and illustrating familiar mathematical content; they present new and interesting ways of approaching familiar tasks.

Some brief notes are appended about some of the major experimental projects. These have been selected because they have all been under way for some years now and so their materials have been tested in the schools.

---

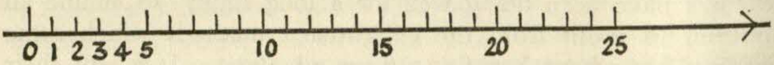
* *Mathematical Thought and the Nature of Learning. The Madison Project View.* Robert Davis. p.4.

While each Project obviously has some unique characteristics, they all share the common goals of emphasising the meaning and structure of mathematics, of adding new, more relevant and significant content, of presenting new approaches to method, of encouraging experimentation and discovery, convinced that children learn mathematics by "doing something about mathematics". In addition to improving the mathematics curriculum, these experimental projects are as much concerned with the problem of finding out more about *how* children learn and about *what* mathematical content is appropriate for different ages and different abilities.

University of Illinois Arithmetic Project

Children learn best when they really want to find an answer and when the answer matters. Accordingly the Project has attempted to develop activities which will stimulate curiosity, exploration, experimentation and discovery, which will present mathematics to children as a fascinating adventure, requiring and deserving hard work, and which at the same time will introduce a great deal of genuine mathematics. One such activity is the Number Line.

Examples of Number Line games are the following:



A "plus" cricket jumps to the right along the Number Line; a "minus" cricket jumps to the left.

(i) If a + 3 cricket begins jumping at 4 and makes two jumps, where will he end?*

---

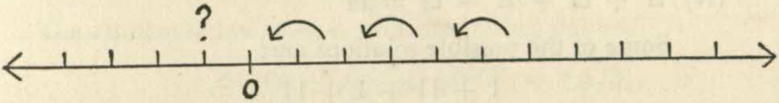
*Many of the new programmes use distinctive symbols, e.g. $^+4$, $^-6$, etc., for positive and negative numbers so as to avoid confusion with the operational symbols for addition and subtraction:

$$^-4 - {}^+9 = {}^-5$$
$$^-3 + {}^-5 = {}^-8$$

Positive and negative numbers can also be distinguished from the symbols for addition and subtraction by writing them $(+4)$, (-6), etc.

(ii) A +4 cricket begins jumping at 5, and after some jumps lands on 25. How many jumps did he make?

Children soon discover that there is no reason why the number line should not be extended to the left, so enabling them to make as many jumps as they wish in either direction. And so they meet up with numbers *below* zero, i.e. negative numbers.



As with most other Projects, considerable use is made of *frames*, e.g.

$$\square + 3 = 12 : \quad 35 = 5\triangledown$$

Children have to fill the correct number into the frames.

These frames play the same logical role as do letters in algebra. They are variables. However, frames have a great psychological advantage over letters in introducing the ideas of simple algebra. When a child sees the letter x he will generally ask "What *is* x?" But when he sees a frame $\square$ he will invariably ask the question: "What can go into $\square$?"

Examples of the kind of problems children solve are the following:

(i) $\square + \square + \square = 18$

The only solution for this number sentence is:

$$6 + 6 + 6$$

(ii) $\bigcirc \times \triangle = 24$

Any of the following solutions are possible.

1×24 : 24×1
2×12 : 12×2
3×8 : 8×3
4×6 : 6×4

Note: There is no limit to the number of additional solutions involving fractions. e.g. $\frac{1}{4} \times 96 = 24$.

(iii) Using only one of the numbers 2, 3 and 4 on each frame, find the largest possible result.

$$\square + \square - \nabla$$

(Ans.: $4 + 4 - 2 = 6$).

(iv) $\triangle + \square + \triangle + \square = 24$

Some of the possible solutions are:

$$1 + 11 + 1 + 11$$
$$2 + 10 + 2 + 10$$
$$3 + 9 + 3 + 9$$

Children are encouraged to solve equations, such as the following:

(a) $\square + \square + 8 = 15$.

What number + the same number $+ 8 = 15$? (Ans: $3\frac{1}{2}$)

(b) $3\,\square + 6 = 30$

(c) $2\,\triangle + 4 = 5$

(d) $4 + \triangle + \square + \triangle + \square = 20$

If the squares are 5, what are the triangles?

Many other matters of logic can be dealt with unobstrusively through the use of frames.

The *commutative* laws for addition and for multiplication are presented through frames:

$$\triangle + \square = \square + \triangle;\ \triangle\square = \square\triangle$$

Generalisations are formed as children try substitutions with whole numbers and even fractions.

$5 + 4 = 4 + 5$ $\qquad$ $3 \times 8 = 8 \times 3$

$\frac{1}{2} + \frac{3}{4} = \frac{3}{4} + \frac{1}{2}$ $\qquad$ $8 \times \frac{1}{4} = \frac{1}{4} \times 8$

The *associative* laws for addition and multiplication are presented in a similar fashion:

$$(\triangle + \square) + \diamond = \triangle + (\square + \diamond)$$

$$(4 + 8) + 3 = 4 + (8 + 3)$$

$$(\triangle \times \square) \times \diamond = \triangle \times (\square \times \diamond)$$

$$(3 \times 4) \times 2 = 3 \times (4 \times 2)$$

The *distributive* law, too, is presented using frames:

$$\diamond\,(\square + \triangle) = (\diamond\,\square) + (\triangle\,\square)$$

$$4\,(3 + 6) = (4 \times 3) + (4 \times 6)$$

$$\tfrac{1}{2}\,(12 + 8) = (\tfrac{1}{2} \times 12) + (\tfrac{1}{2} \times 8)$$

As children work through exercises of this kind they gradually develop the ability to write equations with frames which always yield true statements when numbers are substituted for the frames.

Further particulars about this Project may be obtained from:

The Director,
University of Illinois Arithmetic Project,
Urbana, Illinois, U.S.A.

School Mathematics Study Group (S.M.S.G.)

The proposed S.M.S.G. curriculum is not radically different from the general present-day programme except that more breadth and depth are given to the content: greater emphasis is placed on precision of the language of mathematics, on understanding of the laws and principles of mathematics, and on appreciation of a mathematical system which expands gradually to make more and more mathematical solutions possible. There are, of course, some new topics in addition to new approaches to old ideas. As the S.M.S.G. Committee points out:

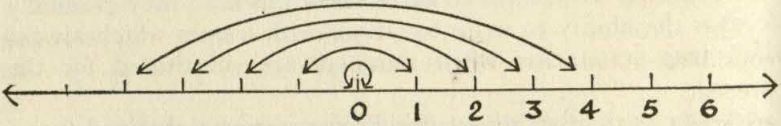
> "The healthy fusion of the old and the new should lead students to a better understanding of the basic concepts and structure of mathematics and provide a firmer founda-

tion for understanding and use of mathematics in a scientific society."*

Among the new topics is the notion of *set*. Through discussion, classroom experiences and exercises on the meaning of set, sets within sets, equal sets, empty set, union of sets, intersection of sets, etc., the children build up a "set language" which will prove invaluable to them later.

Other new topics include intuitive geometry and simple algebraic ideas. Throughout the elementary grades concepts are introduced on an intuitive level by methods which are consistent with later approaches to the topics.

As an example of a somewhat new approach to an old idea, the S.M.S.G. Project introduces the notion of negative numbers by examining the relation—"oppositeness"—between numbers of pairs of dots joined as shown:



Children consolidate their notions of negative numbers by solving simple problems, such as re-arranging a list like:

5 0 −4 6 −3

so that, reading from left to right, each number is *less than* those following it.

Using a simple slide rule, made out of cardboard strips by themselves, children soon acquire facility in the computation of negative numbers.

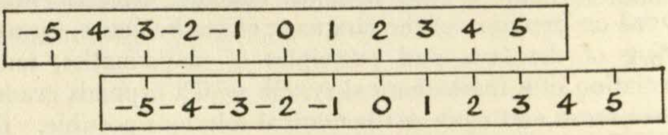


Diagram showing position of slide rule for

$-4 + 5 = 1$.

---

*S.M.S.G. *Mathematics for the Elementary School.* Edward Bros. Inc., Ann Arbor, Michigan. 1961. Foreword.

For subtraction of negative numbers e.g. $6 - (-2)$ S.M.S.G. uses this device. On the analogy that $6 - 2 = ?$ may be solved by asking $6 = ? + 2$ and finding the answer 4, $6 - (-2)$ is solved by asking $6 = ? + (-2)$, resulting in the answer 8.

Throughout its programme, S.M.S.G. insists that insight and skill in computation must go hand in hand in mathematics.

Further particulars about the S.M.S.G. Project may be obtained from:

The Director,
School Mathematics Study Group,
School of Education—Cedar Hall,
Stanford University, Stanford, California, U.S.A.

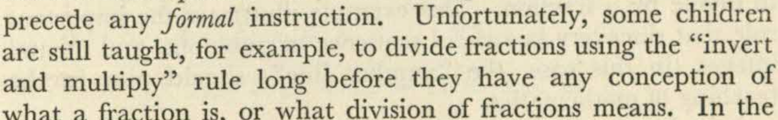
STATE INSTITUTE OF EDUCATION ★ BANIPUR ★

The Madison Project

This Project very rightly assumes that for young children considerable *experience* with basic concepts and techniques should precede any *formal* instruction. Unfortunately, some children are still taught, for example, to divide fractions using the "invert and multiply" rule long before they have any conception of what a fraction is, or what division of fractions means. In the Madison Project approach, children would have several years of informal experience with fractions *before* arriving at the "invert and multiply" rule. Instead of children being *told* how to solve $\frac{1}{2} \div \frac{3}{4}$ the children would be guided to *discover* the solution through a careful sequence of problems such as the following:

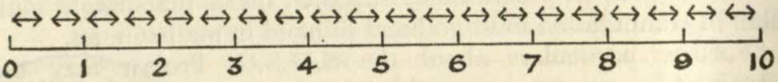
Children would be asked to interpret, say, $10 \div 2$ on a "number line" or with the use of Cuisenaire rods.

They will soon discover that there are five 2's in 10.



Next the children would be asked to interpret, without being told what to do, $10 \div \frac{1}{2}$, i.e. "how many times does $\frac{1}{2}$ go into 10"?

Again they will readily discover from pictures such as this that there are twenty $\frac{1}{2}$'s in 10, i.e. $\frac{10}{\frac{1}{2}} = 20$.



Without any formal instruction, children soon perceive that

$\frac{10}{\frac{1}{3}} = 30$; $\frac{10}{\frac{1}{4}} = 40$ etc.

By solving progressively more complicated examples, e.g. $\frac{\frac{1}{2}}{\frac{1}{4}}$ ("How many times does $\frac{1}{4}$ go into $\frac{1}{2}$?") children who, by now, should have become accustomed to look for and to discover patterns, will work out *for themselves* a complete "theory" of how to divide by a fraction. This example illustrates the important role that discovery, a careful sequence of questions, and mental images (in this case, the "number line") should play in the teaching of mathematics.

As in the case of fractions, the Madison Project activities provide for long informal experience with *equations*, indeed, with nearly every basic concept and technique, before any formal consideration is given to these topics.

The Project directors point out that by "experience" they do *not* mean "concrete experience". In fact, most of the Project's activities deal with abstract matters. The directors defend this action by affirming that "this sometimes seems incongruous to adult observers, but it doesn't seem so to children".

The Project work books almost never tell the children what to do, or how to do it. Instead they ask questions. By *understanding* and *thinking through* the questions on their own, children are stimulated to discover the basic concepts of mathematics and to think creatively about mathematics. Here is an example of the sort of creative experiences that the Project encourages as against the traditional telling children what to do.

Children will have found out for themselves what is meant by a false sentence, an open sentence and a true sentence, e.g.

(i) FALSE SENTENCE

New York is the capital of England.

OPEN SENTENCE

..........is the capital of England.

TRUE SENTENCE

London is the capital of England.

(ii) $4 + 5 = 11$ $\square + 5 = 9$ $4 + 5 = 9$

In (i) (London) is the truth set; in (ii) (4) is the truth set.

The use of open sentences makes it possible to make general statements about numbers. By this procedure children discover identities. Identities are sentences which remain true regardless of what numbers are substituted, e.g.

$$\triangle + 9 = 9 + \triangle; \qquad \square + 0 = \square$$

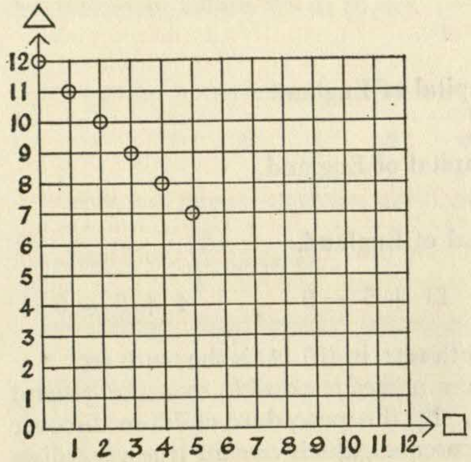
$$\square + \square + \square + \square = 4 \times \square$$

$$3 \times (\nabla + \square) = (3 \times \nabla) + (3 \times \square)$$

Suppose the children are given the open sentence $\square + \triangle = 12$ They will have no difficulty in making a truth set table, i.e. a table of pairs of numbers that will make the statement true.

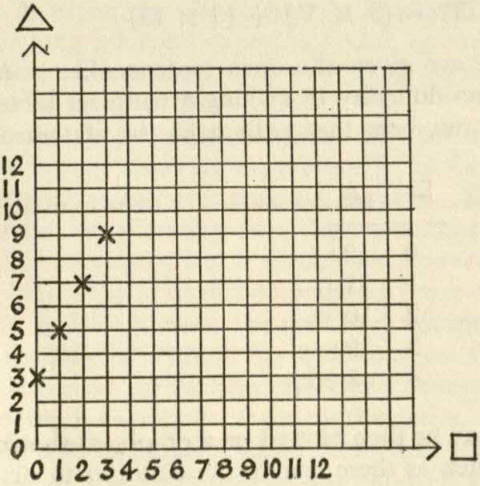
$\square$	$\triangle$
0	12
1	11
2	10
3	9
4	8 etc.

These number pairs may be then plotted on a graph, as shown on page 100. Graphs such as these enable the children to discover various interesting patterns. Many children prefer to review the "addition facts" through graphs such as these. (They will soon realise when a wrong pair is plotted!) The graph confirms the correct answers of children, and so saves them from having to ask the teacher.



Once children have the added security of the pattern on the graph, they will go beyond whole number addition facts to fractions and even negative numbers.

When the number-pairs of the truth set for the equation $\triangle = (2 \times \square) + 3$ are plotted on a graph, the children will recognise the pattern and discover the progression by two.



The slope and intercepts of linear graphs are "discovered" as patterns. The Madison Project considers that this *ability to discover patterns in abstract material* is one of the most essential mathematical skills—possibly *the* most essential skill.

Professor Robert B. Davis, Director of the Madison Project, says: "What we want to do with our children then is to give them a situation where they can try something, then observe the results, evaluate the results, make tentative hypotheses about what works and what does not work, and gradually build these hypotheses into a method for getting the desired results with every new trial."

Matrices, simultaneous and quadratic equations, graphs, mensuration, formulae, etc., are good foils for the kind of teaching Professor Davis looks for. Accordingly, these topics are taught, not for their utility, but because they will provide children with some experience of *genuine mathematical discovery.*

In the case of simultaneous equations, such as the following:

$$\square = (3 \times \triangle) + 2$$

$$\square = \triangle + 10$$

the children experiment to find replacements for the frames so that the *two* equations will be solved. (Of course, □ and △ must be different numbers; and the same replacement must be used for both squares and for both triangles.)

Further particulars about the Madison Project may be obtained from:

The Director,
Mathematics Department,
Syracuse University,
Syracuse 10, New York, U.S.A.

The Greater Cleveland Mathematics Programme (G.C.M.P.)

Extensive use is made of the discovery approach to learning. Children are led, through skilful teaching, to experience the thrill of discovering for themselves the fundamental laws and principles that are part of the logic and structure of mathematics.

The commutative, associative and distributive laws of mathematics are used to give meaning to the processes of addition and multiplication.

The *commutative* property becomes obvious to children when they realise from the addition tables (or matrices) they have constructed that 2 + 4 = 4 + 2: 3 + 4 = 4 + 3, etc.

+	1	2	3	4	5
1	2	3	4	5	6
2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10

(They will also note that the "double" numbers lie on the diagonal from the top left to the bottom right corners of the table.)

This same table will enable children to appreciate the inverse relationship of subtraction and addition. If a child is solving 7 − 4 = □, having located the sum (7) in the body of the table, he will have no great difficulty in "discovering" the missing addend.

Children who know the addition combinations to 10 and who have an understanding of the *associative* (or regrouping) property are easily able to master addition combinations above 10, e.g.

$$8 + 6 = 8 + (2 + 4) = 10 + 4 = 14.$$

×	1	2	3	4	5
1	1	2	3	4	5
2	2	4	6	8	10
3	3	6	9	12	15
4	4	8	12	16	20
5	5	10	15	20	25

As facts are explored and discovered they are recorded in addition and multiplication tables for later use in discerning relationships. (Teachers are repeatedly urged to teach for relationships.) When children construct and explore a multiplication table such as this, they will "discover":

(i) the identity element for multiplication (i.e. 1 × any number yields the number);

(ii) the commutative property, viz. $3 \times 5 = 5 \times 3$;

(iii) the patterns of multiples for each factor found in the top row or in the left-hand column;

(iv) the diagonal drawn from the top left to the bottom right corner gives the square of each factor;

(v) the inverse relationship of division and multiplication.

Multiplication is also presented as successive addition of an addend and division as successive subtraction of a factor, e.g.

$3 + 3 + 3 + 3 \quad = \quad 4 \times 3 \quad = \quad 12$

Three is added 4 times, therefore $4 \times 3 \quad = \quad 12$

Three can be subtracted from 12 four times,

Therefore $12 \div 3 = 4$

$$\begin{array}{rl} 12 & \\ -3 & \text{1st time} \\ \hline 6 & \\ -3 & \text{2nd time} \\ \hline 9 & \\ -3 & \text{3rd time} \\ \hline 3 & \\ -3 & \text{4th time} \\ \hline 0 & \end{array}$$

Children are guided to understand that the property of commutativity does not apply to subtraction and division, e.g.

$12 - 4$ is not equal to $4 - 12$;
$20 \div 5$ is not equal to $5 \div 20$.

Nor does the associative (regrouping) property apply, e.g.

$(9 - 6) - 2$ is not equal to $9 - (6 - 2)$;
$(24 \div 8) \div 2$ is not equal to $24 \div (8 \div 2)$

When the children apply the associative property to multiplication they will be encouraged to reason as follows:

$$40 \times 7 = (10 \times 4) \times 7 = 10 \times (4 \times 7) = 10 \times 28 = 280$$

The use of the *distributive* property will assist children to simplify multiplication as follows:

$$7 \times 9 \quad = (7 \times 10) - 7 = 70 - 7 = 63$$

$$6 \times 34 = (6 \times 30) + (6 \times 4)$$
$$= 180 + 24 = 204$$

$$124 \times 63 = (100 + 20 + 4) \times (60 + 3)$$
$$= 100 \times (60 + 3) + 20 \times (60 + 3) + 4 \times (60 + 3)$$
$$= 6000 + 300 + 1200 + 60 + 240 + 12$$

Further particulars about the G.C.M.P. may be obtained from:

The Executive Director,
Greater Cleveland Research Council,
75 Public Square,
Cleveland 13, Ohio, U.S.A.

Conclusion

In these few pages a necessarily brief account has been given of the many new and exciting developments taking place in the teaching of mathematics at the elementary level. These developments arose from the nationwide concern for more effective learning and appreciation of elementary mathematics. The various experimental projects, concerned with the total programme of finding out more about how children learn as well as with improving the mathematics content, have certainly added new dimensions and new directions to the elementary school mathematics programme. Even at this moment further experiments are in progress to develop even better ways of preparing today's young children for the future—a future to be dominated increasingly by the influence of mathematics and science. It is this never-ending search for better and better techniques and content that makes mathematics teaching so challenging and exciting.

Appendix B

Post-War Developments in the Teaching of Mathematics in the Primary Schools of the United Kingdom.

Introduction

The post-war period has seen some striking changes not only in the content and method of mathematics teaching in the primary school but also in the attitude of teachers and others to the subject. The adult who openly admits an ignorance of mathematics is becoming rarer. We have, however, still a long way to go before the subject achieves anything approaching the same status as the study of English. Transfer from the primary school to the different types of secondary school in the United Kingdom at the age of 11 to 12 has been largely determined by tests of ability and of attainment in English and arithmetic.

Selection for Secondary Education

Entry to the academic grammar schools (the main route to the university) has often been weighted in favour of the child who showed linguistic ability rather than mathematical insight. This factor, combined with the elementary nature of the tests used, has tended usually to restrict the mathematical curriculum in the primary school. There are welcome signs that this situation is changing. Many local education authorities (especially in the larger county areas) are reducing the formality of the procedure for entrance into secondary education and relying more on teachers' estimates of the children's needs. This has meant that primary schools have been free to experiment and to widen the scope of their teaching without fear that their children might be at some disadvantage compared with children in schools where the emphasis is still on formal rote-learning and the acquisition of mechanical skills.

Mathematics and not Arithmetic

One direction in which teachers have sought to try out new ideas has been in the teaching of mathematics. A few years ago we would have seen "Arithmetic", "Number", possibly "Sums" or even "Computations" on the time-table but now "Mathematics" is the rule rather than the exception.

The turn of the tide was really marked by the publication of *Teaching of Mathematics in Primary Schools** by the Mathematical Association in 1955:

> "At first sight the word Mathematics may seem pretentious when used of education in the Primary School. The words Number and Arithmetic, familiar in syllabuses and time-tables, are in fact, equally so, and they have the disadvantage of suggesting limitation of aspect rather than elementary treatment. . . . Just as we have stressed the wholeness of a child's growth, so also we emphasise that progress towards Mathematics must not be limited by a narrow concern with computation as the only field of experience."†

The Report had an immediate impact throughout the country and there was an upsurge of interest in primary schools in re-thinking the content of the curriculum. To some extent, the move to introduce some of the elements of scientific method into the primary school complemented this development and one tended to progress hand in hand with the other. Most areas of the United Kingdom experienced a demand for teachers' courses with which it was difficult to keep pace and various schemes were tried out which would spread the opportunity for teachers to learn of new ideas and techniques as widely as possible.

The Provision of Teachers' Courses

Her Majesty's Inspectors did a great deal at this time by organising a series of conferences and courses up and down the country. The Kent Education Committee, for instance, in addition devised a scheme of training for "Mathematical Missionaries". It was intended that an élite of secondary school specialist teachers should combine with interested primary school head teachers to set up local groups of primary school teachers who would be guided and helped in "Discovering Mathematics" activities at weekly meetings. Since this scheme began, nearly a thousand of the five thousand primary school teachers in Kent have taken part in these courses. Naturally there remain today very few of the County's 700 primary schools where the new ideas have not yet penetrated.

---

*Published by G. Bell & Sons Ltd. 121 pp.

†*Ibid.* page (vii).

Mathematics of the Environment

The fundamental principle behind all this effort has been that children themselves should explore their environment under the guidance of their teachers. It is the teacher's task to provide the learning situation and to facilitate the mathematical education of the child through first-hand experience.

> "The importance of experience as the necessary foundation for the formation of those concepts and patterns of thinking is stressed throughout. . . . Practice without the power of mathematical thinking leads nowhere; the power of mathematical thinking without practice is like knowing what to do but not having the skill or tools to do it; but the power of mathematical thinking supported by practice and rote-learning will give the best opportunity for all children to enjoy and pursue Mathematics as far as their individual abilities allow."*

This was the theme of Miss E. E. Biggs, H.M.I., who was largely responsible for a national series of residential and day-time conferences and courses. Teachers heard lectures on present-day trends and individual speakers discussed various aspects of the mathematical curriculum. The main feature of these meetings, however, lay in the learning experiences that the teacher-students themselves were expected to undertake. They (literally) worked their way through three large rooms in which had been set out instructions for conducting a wide variety of exploratory investigations. For example:

(i) They might be provided with a model of a telephone kiosk and asked (a) to find out its scale, (b) to draw a graph showing relationships between corresponding measurements on the model and on an actual kiosk, (c) to compare corresponding areas, and (d) to compare corresponding volumes.

---

*Ibid. page 4.

(ii) "Time the model train as it makes a circuit of the track. Devise an experiment to find its speed in feet per second and in miles per hour."

(iii) "A householder complained that a ton of coke delivered to him during wet weather contained 15% of water. Find out whether this is possible."

As a consequence of these unusual, often amusing, but always demanding tasks, teachers returned to their schools determined to try out these "Discovery" techniques in exploring the environment. This meant that mathematics was becoming more integrated with the science teaching which in turn had a much wider range than the mere study of Nature. Inevitably, the classroom began to take on a new look as children extended their activities into the outside world. There is now an emphasis on the use of the graph to summarise and to investigate the nature of a variety of relationships. Catalogues of mail-order firms have become a rich source of material: "How does the cost of a product fall as the quantity bought increases?" is a question rich with possibilities.

Spatial Mathematics

Similarly, much of this "New Mathematics" involves an appreciation of spatial relationships. In this respect, many educationists feel that there is more mathematics (in the real sense of the word) taught in the infant school (age-range 5 years to 7 years) than in the junior school (7 to 11). They feel that the junior school child is made to reach for pen and paper too often and at too early a stage.

> "The very use of language in describing what is seen and done brings out geometrical ideas, gives them clarity, and makes them familiar, but teachers should beware of letting their pupils acquire verbal facility in advance of observation: it cannot be too strongly emphasised that the essence of geometrical study at the Junior stage must be the children's

own active operations building up an understanding which leads to more effective constructional work, to greater certainty and accuracy in such work, and to pleasure in the consistency and cohesion which they discover in the spatial properties of diverse objects."*

A corollary of this view is the approach to geometrical ideas through the solid to the flat. Pyramids, cones and boxes will be handled and experienced before the triangle, the circle and the square are seen as patterns on a two-dimensional piece of paper.

Although everyday life provides the background and yields much of the raw material for the lesson, the aim is by no means purely utilitarian. We build towards the abstract from the concrete and always seek to abandon the aids to learning when their use is no longer necessary for understanding. There is no purpose in copying slavishly a treatment which has been successful elsewhere; a careful regard must be paid to its relevance in the teacher's own scheme and to the specific level of ability and attainment of the individual child. Equally there is no value in insisting that the bright child should always learn from actual experience. He may easily perceive, for instance, the relationship between the weights of two oranges whose diameters are in the ratio of 1:2 without recourse to verification if he has already built up a rich fund of experience.

The Structural Approach to Mathematics

Perhaps the most significant feature of the post-war years has been in the rapid development of the structural approach to number concepts, but it must be remembered that this kind of apparatus has been in use for many years.

(i) *The contribution of Montessori.* Madame Montessori towards the end of the last century was advocating the breaking down of arithmetical operations into separate skills. Here was a very significant influence indeed on the primary school in Europe. She devised different pieces of apparatus for the different steps in a mathematical process. For example, large figures cut out in sandpaper allowed a child to learn the shape of the written figure but "stairs" and skittles developed an understanding of number relationships. Hers was a philosophy which condemned the "make-believe" but encouraged realism in mathematical experience. As with all pioneers, her methods today suffer from a comparison with what has followed and much of what she taught may seem rather too academic for today's educational climate.

---

*Ibid page 80.

(ii) *The influence of Piaget.* Perhaps Piaget has had the greatest influence of all in shaping the mathematics teaching of the immediate past generation not only in this country but also in much of Western Europe. Yet it must be remembered that he was not primarily concerned in his own native country of Switzerland with the educational implications of his ideas. Broadly, he believed that the child developed intellectually by clearly marked steps as he perceived fundamental truths. Many educationists, (for instance, Susan Isaacs) are reluctant to concede that development is not continuous. They believe that the thought processes of children and of adults differ in degree and not in kind. Piaget's main hypothesis is that the structure of number concepts, for example, is matched by the development of logical thought. A child's experience may be modified, he believes, to achieve an optimum rate of development. The child needs an analogy as a basis for his reasoning. He makes further progress in his learning when he comes across the unfamiliar. When this does not fit into the framework of his analogy, his analogy needs to be extended or modified. How much the child is helped to do this lies in the difference between enlightened teaching and restrictive coaching. In the initial stages the child appreciates ideas of "greater than", "equal to" and "smaller than" and finally (Piaget suggest at about the age of 7) he becomes aware of the principle of constancy or of conservation. Certainly, discussion of Piaget's beliefs has made teachers question more fully what they were teaching and how they were teaching. There is now a tendency to question whether the assumptions we have made in the past, for instance, about what is obvious to a child, are really found to be true in practice.

(iii) *The main types of structural apparatus.* These psychological developments in Europe have paved the way for the widespread use of structural apparatus in primary schools (and to a lesser extent in secondary schools) in the United Kingdom. Although Gattegno was advocating the use of Cuisenaire rods in the immediate post-war years, it was not until the 1960s that their use became a signficant feature of primary school work. Another similar system, "Colour Factor", incorporating two more rods (for 11 and 12) has also been put on the market.

Alongside this development the Stern apparatus was increasingly finding a place for itself but more particularly in infant schools and in the special schools for educationally sub-normal pupils. Finally, a great deal of interest centres around the M.A.B. (multibase arithmetic blocks) of Dienes and his A.E.M. (algebraic experience material).

To a large extent, the question of cost has been responsible for the differences between the respective frequencies of use. The three main types of structural apparatus are not really in competition with each other. Although they have some common elements they cover different mathematical concepts. The decision which faces the teacher is not really: "Which will be of the greatest help to my children in developing skill in number work?" but rather: "Which aspect of number work do I wish to illustrate?" In other words they all have a great deal of value if used at the right time with the right children.

(iv) *Cuisenaire.* See Appendix C.

(v) *The Stern apparatus.* The Stern apparatus consists of a wider variety of material. Much of it is self-teaching and children are easily led by the nature of the structure to understand number-pattern relationships and the basis of many operational functions. The larger rods (of ¾ in. section instead of 1 cm.) enable infants and children of low levels of ability, with poor muscular control to handle the rods much more comfortably. Colour differences have no intrinsic significance but one drawback is that the marking of the rods in unit steps encourages one-by-one counting. The Stern material does not call for the same degree of skill from teacher and, as methods tend to be more circumscribed there is less opportunity for the teacher to experiment.

(vi) *Dienes Apparatus.* The Dienes M.A.B. consists of wooden "units", "longs", flats" and "blocks" in a variety of different bases: 3, 4, 5, 6 and 10. In the base 10, for instance, the number 3,274 could be represented by 3 "1,000 - blocks" + 2 "100 - flats" + 7 "10 - longs" + 4 "units".

His A.E.M. consists of wooden triangles, prisms, strips, squares, balance with hooks on arms, ring weights, pegboards, etc. All this is likely to appeal to the 10-year old and the secondary school child rather than to the younger infant. Much of its value depends on the assignment sheets devised by Dienes.* The apparatus places great stress on the development of basic mathematical concepts, in particular in the algebraic discipline.

Few teachers are equipped by education and training to obtain the maximum value from using the material and not many teachers or educational advisers have yet built up a sufficiently large fund of experience which they could pass on to others seeking guidance. One exception is perhaps to be found with G. W. Sealey (Adviser for Schools in Leicestershire) who has had a significant influence on the mathematics curriculum of schools in the Midlands. It still is the case, therefore, that not many teachers really understand what Dienes is about and there is a tendency to condemn the apparent complications and to overlook the simplicities beneath the surface. Dienes (like Gattegno) places great emphasis on the value of preliminary play with the material. This needs to be introduced at the right stage of the child's development and to be followed by some formalisation of play. Constructive activities with the apparatus precedes analysis of the principles involved. Finally, he sees a need for as wide a variety as possible of experiences with different kinds of material in order to emphasise the invariability of the mathematical principle and to practise the new ideas. He sees little need for building up a rigorously logical and interlocking system of mathematical properties.

(vii) *Structural Apparatus—Summing up.* The good teacher will make use of all this structural apparatus at some time or other (school funds permitting!) but a complete course of mathematics cannot be based on structural apparatus alone. For instance, it is not easy to extend the concept of number to include directed numbers without the use of a linear scale.

---

*Formerly of the University of Leicester.

These mathematical "concrete analogies" need to be linked with all aspects of environmental or social mathematics. They do not provide an easy road to success for the teacher. Many teachers who had turned, for instance, to Cuisenaire-type rods in the hope that they would do their work for them, have found to their disappointment that their own intellectual contribution has needed to be very much greater than they had expected. On the other hand, the return in the shape of pupil's greater understanding has increased appreciably and in direct proportion to the magnitude of the teacher's own efforts. A word of warning needs to be uttered:

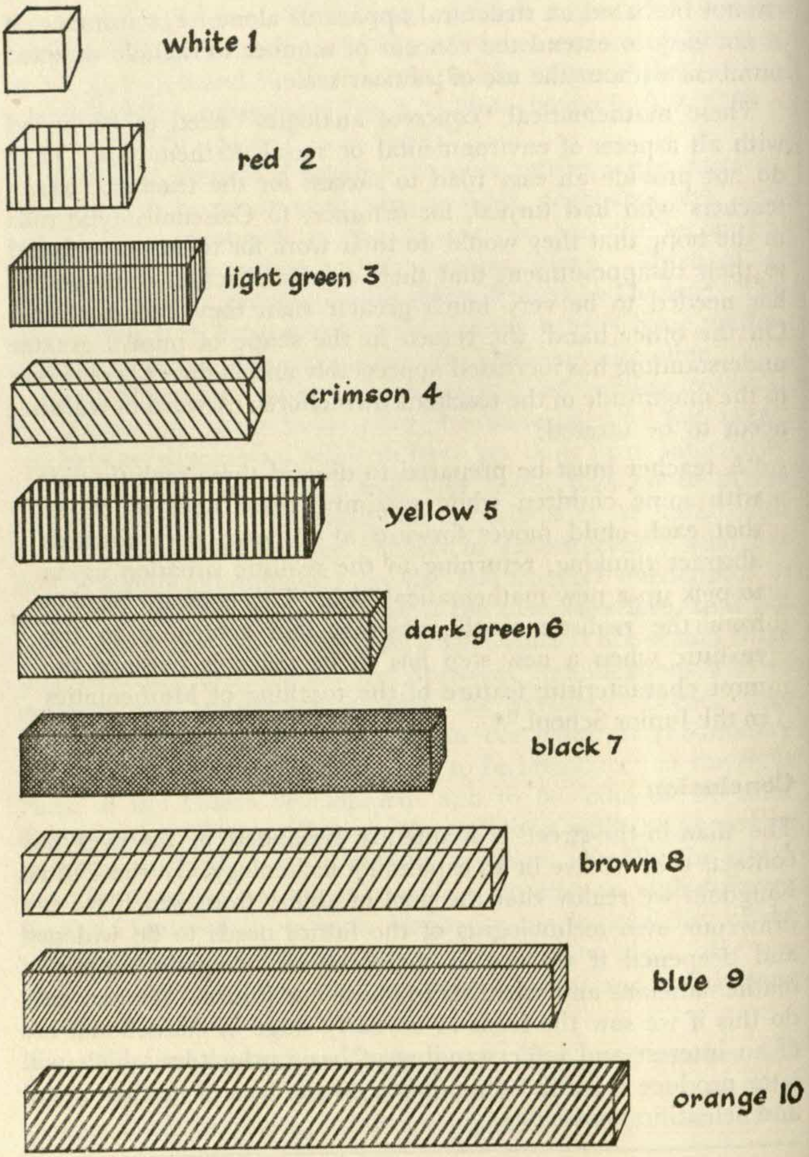
> "A teacher must be prepared to discard these realistic aids with some children while retaining them with others so that each child moves forward at his own pace towards abstract thinking, returning to the realistic situation again to pick up a new mathematical idea. This constant passing from the realistic to the abstract and returning to the realistic when a new step has to be taken is perhaps the most characteristic feature of the teaching of Mathematics in the Junior School."*

Conclusion

The man-in-the-street is frequently reminded in his everyday contacts that we live in an expanding technology. In the United Kingdom we realise that the pool of ability from which we can draw our own technologists of the future needs to be widened and deepened if we are to maintain an adequate supply of mathematicians and scientists to guide our future. We can only do this if we sow the seeds at an early stage in educational life of an interest and understanding of basic principles which will later produce a fruitful abundance on the tree of mathematical and scientific knowledge.

---

*Ibid. page 94.


white 1
red 2
light green 3
crimson 4
yellow 5
dark green 6
black 7
brown 8
blue 9
orange 10

Appendix C

The Cuisenaire Method

Among the new techniques used with great success in Europe, not only in the Primary school but also in Secondary schools, is the Cuisenaire Method. Of course, some very exaggerated claims are made for the method, for instance, that "algebraic equations which are not common in university courses can be solved by eleven year olds." Admittedly adults can use the technique to solve algebraic equations but that is very different from saying that an eleven year old can do likewise. But there is no doubt that this new technique facilitates mathematical working and, more important, makes children familiar with mathematical *concepts*. It will come to be used increasingly in schools in future.

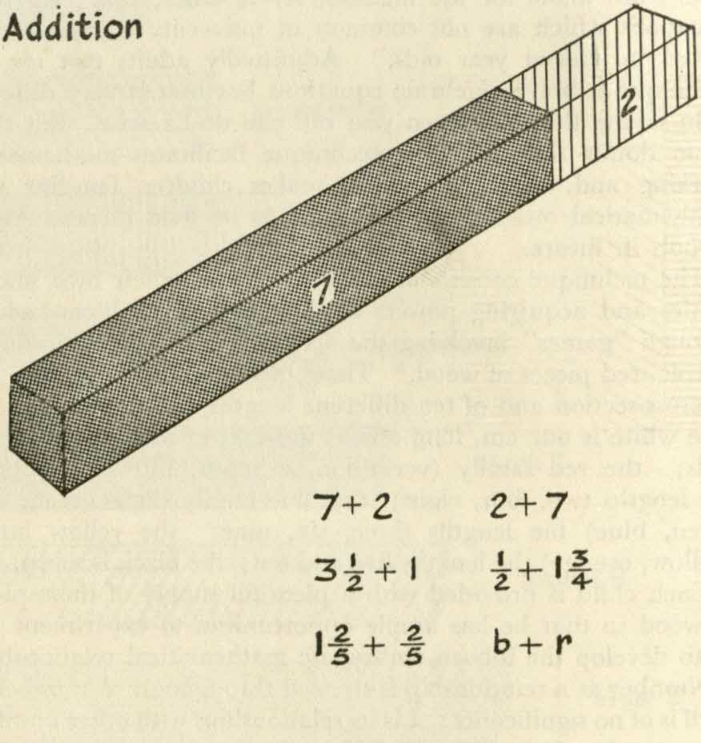
The technique consists of children creating their own mathematics and acquiring powers of thought and problem-tackling through "games" involving the arrangement and manipulation of coloured pieces of wood.* These pieces of wood are 1 sq. cm. in cross-section and of ten different lengths, coloured as follows. The white is one cm. long and is the unit of measure of all the rods; the red family (vermilion, crimson, brown) comprises the lengths two, four, eight; the blue family (light green, dark green, blue) the lengths three, six, nine; the yellow family (yellow, orange) the lengths five and ten; the black is seven.

Each child is provided with a plentiful supply of these pieces of wood so that he has ample opportunities to experiment and so to develop the inborn embryonic mathematical relationships.

Number as a relationship is stressed throughout. A number by itself is of no significance; it is its relationships with other numbers that matters. It is not a rod which represents a number, but the relationship of one rod with another that is of vital significance. Even the five year old will recognise that a crimson rod (4) and a dark green rod (6) give the same length as an orange rod (10), and that dark green and crimson are also equivalent to orange. These young children will not only become familiar with groups of equal numbers, but with groups that are "bigger than" or "smaller than" other groups. Moreover, they will later come to realise intuitively that an addition sum such as " 2s. + 8d." or "\$2 + 8c" is quite different from the addition sum "2 + 8."

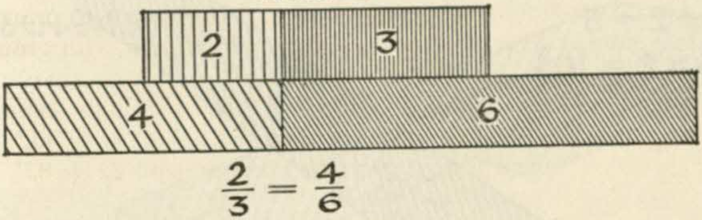
---

*Obtainable from the Cuisenaire Co., 11 Crown Street, Reading, England.



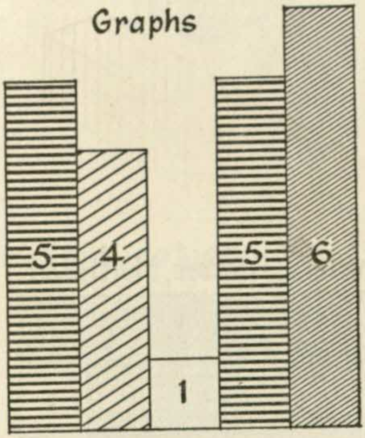
By fitting the rods end to end, side by side, on top of each other to form larger solids, etc., the child becomes familiar with such fundamental mathematical concepts as addition, subtraction, multiplication, division, graphs, fractions, factors, multiples, ratio, area, etc.

Ratios

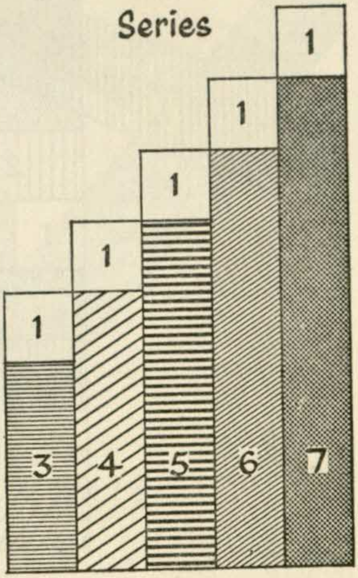


$$\frac{2}{3} = \frac{4}{6}$$

Graphs

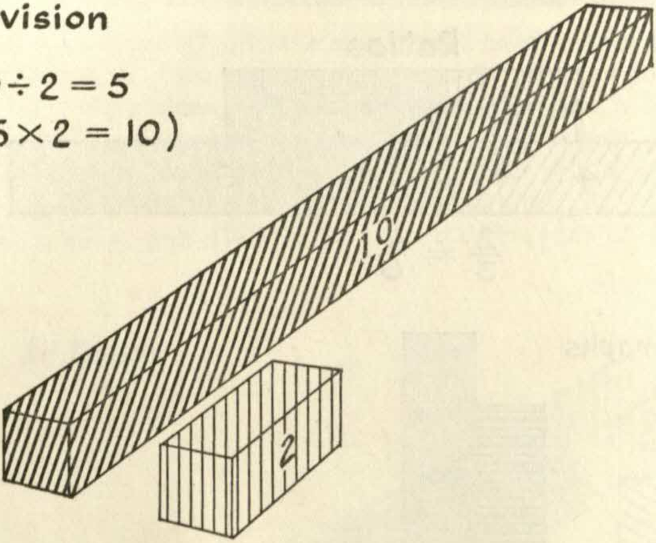


Series

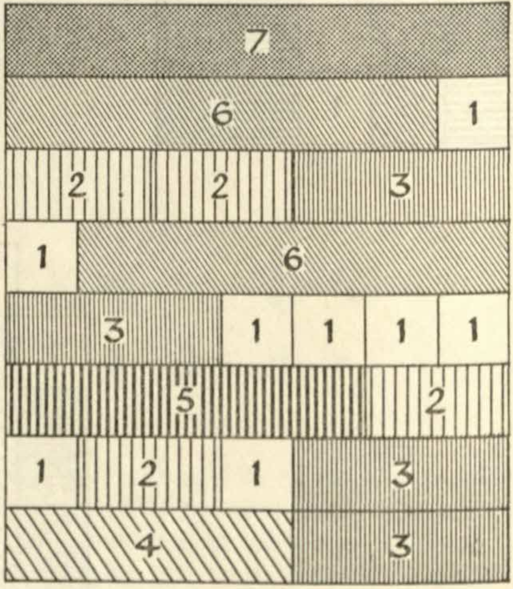


Division

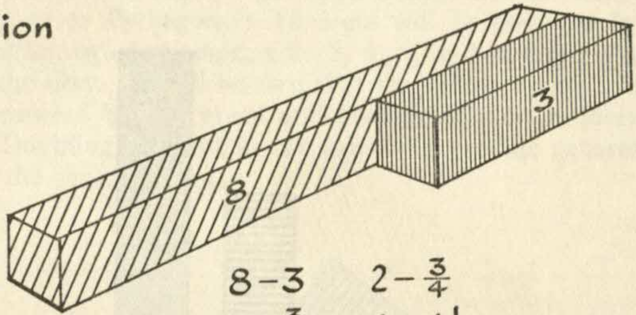
$10 \div 2 = 5$
$(5 \times 2 = 10)$



Number Dynamics, Factors, Multiples.



Subtraction

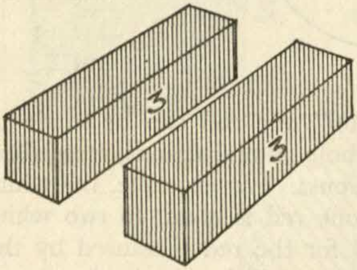


$8-3$ $2-\frac{3}{4}$

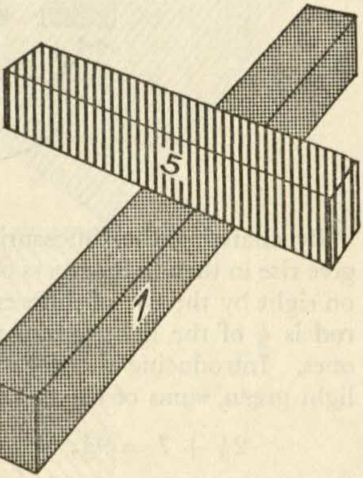
$1-\frac{3}{8}$ $4-1\frac{1}{2}$

$A-B$

Multiplication



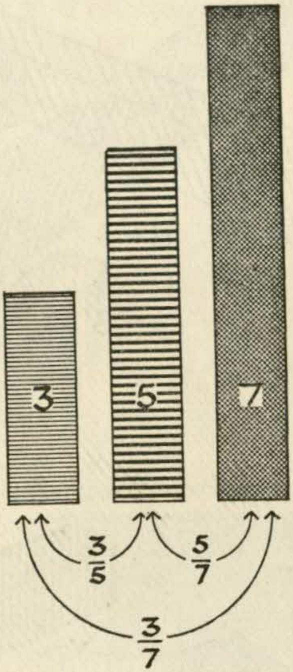
$2\times3=6$



5×7 7×5

$1\frac{1}{4}\times1\frac{3}{4}$

Fractions

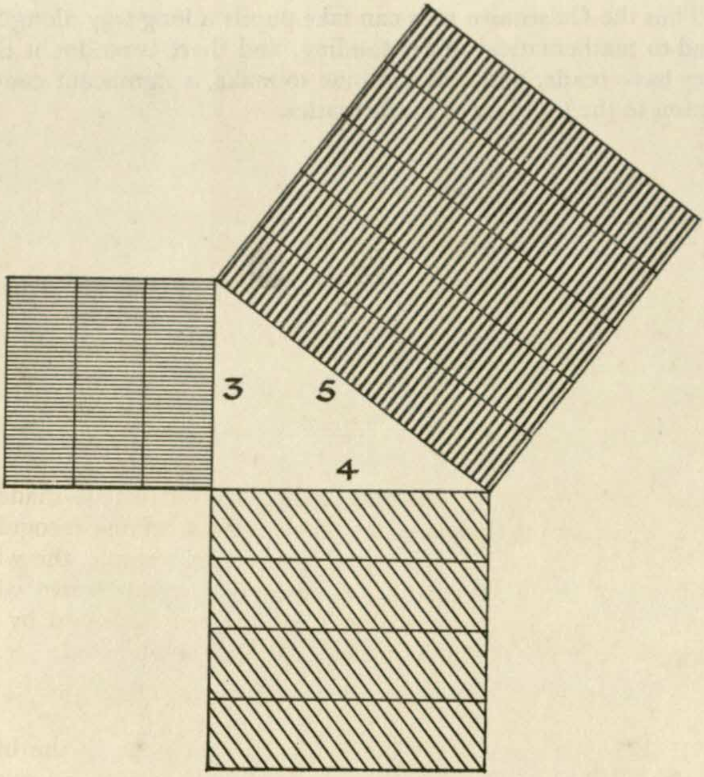


By changing the "measuring rod" one rod can be made to give rise in turn to members of a whole set of fractions, recognised on sight by the use of different colours. For example, the white rod is $\frac{1}{3}$ of the light green, and one red is equal to two white ones. Introducing the notation $\frac{2}{3}$ for the red measured by the light green, sums of the following type will be obtained:

$$2\tfrac{1}{3} + 7 = 9\tfrac{1}{3};\ 4 + 4\tfrac{2}{3} = 8\tfrac{2}{3};\ 5\tfrac{2}{3} + 6\tfrac{1}{3} = 12$$

If the red measures the black the fraction is $\frac{7}{2}$; if the black measures the red, it is $\frac{2}{7}$. Similarly, a light green rod over a black represents $\frac{3}{7}$, black over light green $\frac{7}{3}$. If the red measures the light green and the crimson the dark green, the result is the same $\frac{2}{3}$ or $\frac{4}{6}$.

The Cuisenaire rods have wide application. For example, an introduction to Pythagoras's Theorem will be obtained by forming a right-angled triangle, sides 3, 4, 5, and building the squares on the sides. It will be seen that the yellow square can be exactly covered by the green and crimson squares suitably arranged. Doubling or trebling the sides will provide squares that satisfy the same relationship.



Moving from the concrete to the abstract is an important step. If the use of the rods be implicit, rather than stated, the work becomes more algebraical in nature, and the children, though not aware of the explicit algebraic notation, will become aware of those mathematical concepts as are expressed in such algebraic statements as:

$$\frac{a + x}{b + y} \qquad \frac{2xv}{3} \quad \text{etc.}$$

Thus the Cuisenaire rods can take pupils a long way along the road to mathematical understanding, and there is no doubt that they have made, and will continue to make, a significant contribution to the teaching of mathematics.